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A general method for deriving some semi-classical properties of perturbed second degree forms: The case of the Chebyshev form of second kind

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A B S T R A C T

We present a new general method and the corresponding symbolic algorithm *PSDF* for deriving some semi-classical properties of perturbed second degree forms namely: the Stieltjes function, the Stieltjes equation, the functional equation, the class, a structure relation and the second order linear differential equation. We give new explicit results for some perturbed of order 3 of the second kind Chebyshev polynomials.

1. Introduction

The general method and the algorithm *PSDF* presented in this work are based on the algebraic approach of orthogonal polynomials introduced by Pascal Maroni mainly in Refs. [1–6], in particular on an implementation of a set of operations defined in the topological dual space of the vectorial space of polynomials. The basic idea consists in dealing directly with the linear forms, determined by their moments or equivalently by the corresponding Stieltjes series, and their interrelationships, and not with the integral representations of them [5].

Perturbation corresponds to a modification on the first coefficients of the recurrence relation of order two satisfied by orthogonal polynomial sequences. This transformation can promote a deep change of properties; nevertheless there is a large set of forms that are preserved by perturbation: the second degree forms. In other words, the perturbed of a second degree form still is a second degree form. Moreover, a second degree form is also a semi-classical one [5,7]. The general

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method presented herein is based on this crucial fact. We remark that, in general, the perturbed of a semi-classical form is not semi-classical, but a Laguerre–Hahn [5] that satisfies a fourth-order differential equation [8–11]. Laguerre–Hahn forms [12,13] generalize semi-classical and second degree forms. It is worthy to mention here that among the classical forms only certain Jacobi forms are of second degree [14]; from them other second degree forms can be generated by applying several transformations [7,15,14,16]. Furthermore, all self-associated forms are also of second degree [17].

We notice that perturbed orthogonal polynomials have some possible applications [18–20,9,21], which motivate further their study. In fact, during the last years, several authors have worked on this subject considering perturbations of several orders with more or less free parameters with respect to classical, semi-classical, Laguerre–Hahn and others families, studying several properties like generating functions, Stieltjes functions, structure relations and differential equations, separation and the distribution function of zeros and integral representations among others. With respect to the co-recursive case, we would like to cite [22,23,18,24,10], for the co-dilated situation refer to [16,21], for the co-modified [25,26,8], for the generalized co-polynomials see [19,27]. Also, we call the attention to the general Refs. [28,29]. Furthermore, there are some specific works about perturbed Chebyshev families namely [16] on the co-dilated case of the second kind form and [15,30] concerning all the four forms.

It is well known that the four Chebyshev forms [31–33] are the most important cases of second degree forms [15] due to their remarkable properties and utility in applied mathematics, physics and other sciences [34]. In particular, for the purposes of perturbation, the form of second kind is the most simple among them, because it is self-associated, therefore it is often taken as study case in the mentioned literature. So, it seemed important to us to clarify and explicit some semi-classical properties of perturbed Chebyshev polynomials of second kind.

In this work we present a general method, and the corresponding symbolic algorithm, intended to explicit some semi-classical properties of perturbed second degree forms, namely: the Stieltjes function, the Stieltjes equation, the functional equation, the class, a structure relation and the second order linear differential equation. Moreover, we provide the first moments of the perturbed forms. The advantage of this method is its generality: it is intended to work for any perturbation and any second degree form and can be implemented in an algebraic manipulator.

The Chebyshev form of second kind is taken as study example and we give new explicit results for the generalized co-recursive and co-dilated cases of order three. In the same way, other perturbations can be treated and the same procedure can be applied to the other three forms of Chebyshev. This will be the subject of a forthcoming article [35]. Moreover, the characteristic elements presented in this work can be useful in order to obtain other ones like integral representations or make the study of zeros that are crucial in quadrature formulas of numerical integration.

Let us summarize the content of this article. In Section 2, we establish the theoretical framework, we recall the mathematical background necessary to understand the subject of perturbed second degree forms. In particular, we have collected the most important formulas and procedures that compose the general method closely following Refs. [5–7,12–14]. In Section 3, we introduce the method and the algorithm *PSDF—Perturbed Second Degree Forms*. In last section we apply the method step by step to the Chebyshev form of second kind and we give the corresponding new results concerning the above mentioned perturbations. Also, we derive a closed formula for the generating functions of any perturbed Chebyshev family and we compute them in the two treated cases. Notice that often in applications one is interested on numerical concrete values of parameters so that the given formulas will be quite simplified.

2. Theoretical framework

2.1. General definitions and features

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual space. The effect of the ***form*** or ***functional*** $u \in \mathcal{P}'$ on $f \in \mathcal{P}$ will be denoted by $\langle u, f\rangle$. In particular $(u)_n := \langle u, x^n\rangle$, $n \geq 0$, are called the moments of u. Give u, is equivalent to give the sequence of moments $(u)_n$, $n \geq 0$, or the formal series $F\big(u\big)(z) := \sum_{n\geq 0}(u)_n z^n$, or the so-called ***formal Stieltjes function*** [5]

$$S\big(u\big)(z) := -z^{-1}F\big(u\big)(z^{-1}) = -\sum_{n\geq 0}\frac{(u)_n}{z^{n+1}}.$$

Let $f,\ p \in \mathcal{P}$, $u,\ v \in \mathcal{P}'$. By transposition of the operations in $\mathcal{P}$, we have the following operations in $\mathcal{P}'$ [4]. Left-multiplication of a form by a polynomial fu, $\langle fu, p\rangle := \langle u, fp\rangle$. Derivative of a form $u' = Du$, $\langle u', p\rangle := -\langle u, p'\rangle$. Division of a form by a first degree polynomial $(x-c)^{-1}u$, $\langle (x-c)^{-1}u, p\rangle := \langle u, \theta_c p\rangle$, where θ_c is the divided difference operator

$$\big(\theta_c p\big)(x) := \frac{p(x)-p(c)}{x-c}, \quad c \in \mathbb{C},\ x \neq c; \qquad (\theta_c p)(c) = p'(c).$$

Cauchy product of two forms uv, $\langle uv, p\rangle := \langle u, vp\rangle$, where $\big(vp\big)(x) := \langle v, \frac{xp(x)-\xi p(\xi)}{x-\xi}\rangle$ is the right-multiplication of a form by a polynomial.

Let us consider a polynomial sequence $\{P_n\}_{n\geq 0}$ such that $\deg P_n = n$, $n \geq 0$, then there exists a unique sequence $\{u_n\}_{n\geq 0}$, $u_n \in \mathcal{P}'$, $n \geq 0$, called the ***dual sequence*** of $\{P_n\}_{n\geq 0}$ such that $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m \geq 0$.