Journal of Computational and Applied Mathematics 296 (2016) 739–752

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Solving partial integro-differential option pricing problems for a wide class of infinite activity Lévy processes

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ARTICLE INFO

Article history:
Received 8 January 2015
Received in revised form 20 May 2015

Keywords:
Numerical analysis
Partial integro-differential equation
Option pricing
Gauss–Laguerre quadrature
Positivity

ABSTRACT

In this paper, numerical analysis of finite difference schemes for partial integro-differential models related to European and American option pricing problems under a wide class of Lévy models is studied. Apart from computational and accuracy issues, qualitative properties such as positivity are treated. Consistency of the proposed numerical scheme and stability in the von Neumann sense are included. Gauss–Laguerre quadrature formula is used for the discretization of the integral part. Numerical examples illustrating the potential advantages of the presented results are included.

1. Introduction

Since a long time ago empirical observations of the market show the evidence that the price of the underlying asset does not behave like a Brownian motion with a drift and a constant volatility. This fact motivates the emergence of alternative models to the pioneering Black–Scholes model [1]. Alternative models are stochastic volatility [2], deterministic volatility [3], jump diffusion [4–7] and infinite activity Lévy models.

One of the most relevant and versatile Lévy models is the one proposed by Carr et al. the so-called CGMY [8], that belongs to the family of KoBoL models [9]. Apart from these models, other Lévy processes such as Meixner [10,11], Hyperbolic and Generalized Hyperbolic (**GH**) are used to obtain better estimation for the stock returns [12]. The Meixner process was introduced in 1998, it is used when the environment is changing stochastically over the time showing a reliable valuation for some indices such as Nikkei 225 [10].

The generalized hyperbolic distribution was introduced by Barndorff-Nielsen [13] and used to generate Lévy process to capture the real stock price movements of the intraday scale. It is exactly a pure discontinuous behavior of its paths what can be observed [12,14]. Besides that the hyperbolic process is obtained as a special case from the (**GH**) process, it is implemented in various stock markets such as the blue chips of the German market, the DAX and also US stock market showing effective estimation for their returns [15].

However, following [12] the calibration of market option prices shows that depending on datasets, the matching between the actual price and its corresponding estimated value varies from model to another consequently, we cannot say which is the perfect one.

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http://dx.doi.org/10.1016/j.cam.2015.10.027

Table 1
The forms of $\nu(y)$.

Model	The corresponding Lévy measure
KoBoL	$\nu(y) = \frac{C_- e^{-\mathcal{G}\|y\|}}{\|y\|^{1+Y}}\mathbf{1}_{y<0} + \frac{C_+ e^{-\mathcal{M}\|y\|}}{\|y\|^{1+Y}}\mathbf{1}_{y>0}$
Meixner	$\nu(y) = \frac{Ae^{-ay}}{y\sinh(by)}$
GH process	$\nu(y) = \frac{e^{\beta y}}{\|y\|}\left(\int_0^\infty \frac{e^{-\sqrt{2\zeta+\alpha^2}\|y\|}}{\pi^2\zeta\left(J^2_{\|\lambda\|}(\delta\sqrt{2\zeta})+Y^2_{\|\lambda\|}(\delta\sqrt{2\zeta})\right)}d\zeta + \max(0,\lambda)e^{-\alpha\|y\|}\right)$

In this paper we study the option pricing partial integro-differential equation (PIDE) unified model for several Lévy measures $\nu(y)$, given by [16, Chap. 12]

$$\begin{aligned}\frac{\partial \mathcal{C}}{\partial \tau}(S,\tau) &= \frac{\sigma^2}{2}S^2\frac{\partial^2\mathcal{C}}{\partial S^2}(S,\tau) + (r-q)S\frac{\partial \mathcal{C}}{\partial S}(S,\tau) - r\mathcal{C}(S,\tau)\\ &\quad + \int_{-\infty}^{+\infty}\nu(y)\Big[\mathcal{C}(Se^y,\tau) - \mathcal{C}(S,\tau) - S(e^y-1)\frac{\partial \mathcal{C}}{\partial S}(S,\tau)\Big]dy, \quad S\in(0,\infty),\ \tau\in(0,T],\end{aligned} \tag{1}$$

$$\mathcal{C}(S,0) = f(S) = (S-E)^+, \quad S\in(0,\infty), \tag{2}$$

$$\mathcal{C}(0,\tau) = 0; \qquad \lim_{S\to\infty}\mathcal{C}(S,\tau) = Se^{-q\tau} - Ee^{-r\tau}, \tag{3}$$

where $\mathcal{C}$ is the value of a contingent claim, S is the underlying asset and $\tau = T - t$ is the time to the maturity. The Lévy measures $\nu(y)$ are given in Table 1.

Note that the Hyperbolic process is obtained from the **GH** process when $\beta = 0$ and $\lambda = -1$.

To the best of our knowledge, the numerical solution and analysis of Meixner and **GH** models have not been treated. The KoBoL model and in particular the CGMY, see Table 1 with parameter $C_- = C_+$, have been widely studied because it is versatile and includes the finite and infinite activity cases as well as the finite and infinite variation, obtained by changing the value of Yor parameter $Y < 2$. A fairly complete revision of the methods used to solve the CGMY model can be found in [17–20].

In this paper we focus on the numerical analysis of the unified model (1)–(3) for the European case, by proposing a consistent, explicit and conditionally positive and stable finite difference scheme while the integral part is approximated using Gauss–Laguerre quadrature formula. We also include the computation of the linear complementarity problem (LCP) for the American option case using both the projected successive over relaxation method (PSOR) and the multigrid method (MG). The discretization for the differential operator is done using the three-level approximation, while the integral part is discretized as the same as in the European case. So, the integral part of the PIDE operator for the American and European cases is discretized using the Gauss–Laguerre quadrature. Although the three-level method is widely used and it is argued that the approximation error is of order two, however such method has two unsuitable properties, in fact as the method needs the first time step that must be obtained using another method (usually by implicit Euler method), in practice the accuracy is reduced. Also, as it is shown in Example 1 for European option, the three-level method does not guarantee the positiveness.

With respect to previous relevant papers in the field, we should mention the potential advantage of our approach. Apart from the more general unified treatment of a wide class of Lévy models, we do not truncate the integral part for its approximation using Gauss–Laguerre quadrature that reduces the computational cost using a few amount of nodes to approximate the integral and improves the accuracy due to the advantages of Gauss–Laguerre quadrature. An additional positive fact of this approach is that it allows to give error information of the integral approximation as it is shown in Example 4.

The paper is organized as follows. In Section 2, the kernel singularity of the integral part of the PIDE is replaced by adding a diffusion term following the approach developed in [17,18]. Then the reaction and convection terms of the differential part are removed by using suitable transformation as in [20]. Finally in Section 2, the numerical scheme construction is included. Section 3 deals with the numerical analysis of the explicit proposed numerical scheme, including conditional positivity and stability in the von Neumann sense, as well as the consistency. Section 4 is addressed to the study of the American option case, the LCP is solved using the PSOR and MG including the Gauss–Laguerre quadrature discretization for the integral part and the three-level for the differential part. Section 5 includes numerical examples to discuss and validate the results. The Barrier option case is particularly interesting for its application to credit risk problems [21]. In Example 7, we have included the valuation of Barrier option with our approach.

For the sake of clarity, useful integral formula is included. The exponential integral $E_s(\eta)$ is defined by [22, Chap. 5, p. 228]

$$E_s(\eta) = \int_1^\infty t^{-s}e^{-\eta t}dt. \tag{4}$$

2. Scheme construction for European options

Let us begin this section by transforming the PIDE (1) into a simpler one. Since the kernel of the integral in (1) presents a singularity at $y = 0$, a useful technique is to split the real line, for an arbitrary small parameter $\varepsilon > 0$, into two regions