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Strong stability preserving explicit peer methods

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ABSTRACT

In this paper we study explicit peer methods up to order $p = 13$ which have the strong stability preserving (SSP) property. This class of general linear methods has the favourable property of a high stage order. The effective SSP coefficient is maximized by solving a nonlinear constraint optimization problem numerically to high precision. The coefficient matrices of the optimized methods are sparse in a very structured way. Linear multistep methods are obtained as a special case of only one stage.

1. Introduction

The concept of strong stability preserving (SSP) methods was introduced by Shu and Osher [1] for the numerical solution of a hyperbolic conservation law. Discretizing the spatial derivatives with the method of lines (MOL) yields a system of ordinary differential equations

$$y' = f(t, y), \qquad y(t_0) = y_0 \in \mathbb{R}^n, \quad t \in [t_0, t_e]. \tag{1}$$

We assume that spatial discretization is chosen such that the semidiscrete solution satisfies the strong stability property

$$\|y + hf(t, y)\| \le \|y\| \quad \text{for all } y \in \mathbb{R}^n \text{ and } h \le h_E, \tag{2}$$

where $\|\cdot\|$ represents a norm or convex functional. The significance of this condition is that $\|\cdot\|$ will be non-increasing for any approximations computed with the explicit Euler method whenever $h \le h_E$.

Naturally, one is interested in higher order methods satisfying an analogue of (2) for the numerical solution for step sizes $h \le C \cdot h_E$. The positive constant C is called the SSP coefficient of the method. The order of explicit Runge–Kutta methods which preserve strong stability cannot exceed four [2], furthermore their stage-order is only one. Explicit linear multistep SSP methods have no known order bound, however they need a large number of steps for higher order [3]. The deficiencies of classical methods have created a recent interest in high order general linear methods (GLM, [4,5]) which have the SSP property, e.g. [6,7]. In [8] strong stability preserving GLM up to stage order four is considered. Bresten et al. [9] construct multistep Runge–Kutta methods up to order 10.

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In this paper we consider a special class of explicit GLM, explicit peer methods introduced in [10]. These methods have been successfully applied to nonstiff ODEs with step size control in [11]. For these methods the stage order is equal to the order of consistency. We investigate the SSP properties of explicit peer methods and prove a theorem which allows to construct such methods of high order (and consequently of high stage order).

The outline of the paper is as follows:

In Section 2 we introduce explicit peer methods and give a short overview about important properties like consistency, zero-stability and convergence.

In Section 3 we discuss the SSP property for explicit peer methods. We prove a theorem which allows us to determine the SSP coefficient from the parameters of the method. Furthermore a simple relation to linear stability is shown.

The construction of explicit peer methods with large SSP coefficients is presented in Section 4. The numerical optimization problems are solved with Mathematica. We have found SSP methods up to order 13.

Section 5 gives results of numerical tests. We verify numerically the order of the constructed methods, show the advantage of high stage order and illustrate the theoretical SSP properties by computing the step sizes ensuring the TVD property for the Buckley–Leverett equation. Section 6 contains our conclusions.

2. Explicit peer methods

Explicit peer methods for problem (1) as introduced in [10] read

$$Y_{m,i} = \sum_{j=1}^{s} b_{ij} Y_{m-1,j} + h_m \sum_{j=1}^{s} a_{ij} f(t_{m-1,j}, Y_{m-1,j}) + h_m \sum_{j=1}^{i-1} r_{ij} f(t_{m,j}, Y_{m,j}), \quad i = 1, \ldots, s. \tag{3}$$

Here b_{ij}, a_{ij}, c_i and r_{ij}, $i, j = 1, \ldots, s$ are the parameters of the method. At each step s stage values $Y_{m,i}$, $i = 1, \ldots, s$ are computed approximating the exact solution $y(t_{m,i})$ where $t_{m,i} = t_m + c_i h_m$. The nodes c_i are assumed to be pairwise distinct. Defining matrices $B = (b_{ij})_{i,j=1,\ldots,s}$, $A = (a_{ij})$, $R = (r_{ij})$ and vectors $Y_m = (Y_{m,i})_{i=1}^{s} \in \mathbb{R}^{sn}$ and $F_m = (f(t_{m,i}, Y_{m,i}))_{i=1}^{s}$ lead to the compact form

$$Y_m = (B \otimes I) Y_{m-1} + h(A \otimes I) F_{m-1} + h(R \otimes I) F_m,$$

where R is strictly lower triangular.

The coefficients of the method (3) depend, in general, on the step size ratio $\sigma = h_m / h_{m-1}$. Like multistep methods peer methods need also s starting values $Y_{0,i}$. We collect here some results from [10].

Conditions for the order of consistency of explicit peer methods can be derived by considering the residuals $\Delta_{m,i}$ obtained when the exact solution is put into the method

$$\Delta_{m,i} := y(t_{m,i}) - \sum_{j=1}^{s} b_{ij} y(t_{m-1,j}) - h_m \sum_{j=1}^{s} a_{ij} y'(t_{m-1,j}) - h_m \sum_{j=1}^{i-1} r_{ij} y'(t_{m,j}), \quad i = 1, \ldots, s.$$

Definition 1. The peer method (3) is consistent of order p if

$$\Delta_{m,i} = \mathcal{O}(h_m^{p+1}), \quad i = 1, \ldots, s. \quad \square$$

In contrast to explicit Runge–Kutta methods, all stage values of peer methods are approximations of order p to the solution $y(t + c_i h_m)$, i.e., the stage order is equal to the order. This makes these methods advantageous especially for MOL problems when space and time step sizes are reduced simultaneously. By Taylor series follows that a peer method (3) has order of consistency p iff

$$c_i^l - \sum_{j=1}^{s} b_{ij} \frac{(c_j - 1)^l}{\sigma^l} - l \sum_{j=1}^{s} a_{ij} \frac{(c_j - 1)^{l-1}}{\sigma^{l-1}} - l \sum_{j=1}^{i-1} r_{ij} c_j^{l-1} = 0, \quad i = 1, \ldots, s, \ l = 0, \ldots, p \tag{4}$$

is satisfied, [10]. This condition (4) can be written conveniently as

$$\exp(c\sigma z) - B \exp(z(c - \mathbb{1})) - A\sigma z \exp(z(c - \mathbb{1})) - R\sigma z \exp(\sigma z) = \mathcal{O}(z^{p+1}),$$

where $\mathbb{1} = (1, \ldots, 1)^\top$. The exponentials of the vectors are defined componentwise.

The condition (4) for order $l = 0$ is referred to as *preconsistency*. It takes the form

$$B\mathbb{1} = \mathbb{1}. \tag{5}$$

Explicit peer methods are a special class of *general linear methods, GLMs*. GLMs are typically investigated for constant step sizes only. An overview can be found in [4,5]. For the investigation of SSP in this paper we will restrict to the case of constant step sizes $h_m = h$ (cf. [12,8]), too.

A convergence result for peer methods can be found in [11]. For the constant step sizes considered here, the zero stability criterion reduces to the power boundedness of B.