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Some error estimates for the reproducing kernel Hilbert spaces method

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ABSTRACT

In this paper we derive some effective error estimates for the reproducing kernel Hilbert space method applied to a general class of linear initial or boundary value problems. The first error estimate is computable and yields a worst case bound in the form of a percentage of the norm of the true solution which has not yet been discussed according to the knowledge of the authors. The second error estimate is a residual based error estimate, which is expressed in terms of the fill distance, so that convergence is studied for the fill distance tends to zero. This is a generalization and improvement of the existing error estimates. Some numerical results are presented to demonstrate the applicability of the estimates.

1. Introduction

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, learning theory and so on. The reproducing kernels have been successfully applied to several linear and nonlinear problems, see [1–7] and references therein. However, the error analysis for this method has not been thoroughly studied. Recently, some authors have tried to find error estimates of the reproducing kernel method [8,9]. In this paper, we consider the error analysis of reproducing kernel method applied to linear differential equations. A nonlinear problem is usually solved by an iterative scheme; at each iteration the problem is approximated by a linear one, and thus the core calculation is similar in both cases, so the error analysis of the method for linear problems is therefore relevant also to nonlinear problems. We study an error estimate which is computable and yields a worst case bound in the form of a percentage of the norm of the true solution which has not yet been discussed according to the knowledge of the authors. Another error estimate is a residual based error estimate, which is expressed in terms of the fill distance, so that convergence is studied for the fill distance tends to zero. Our aim is to generalize and improve the error estimates presented in [8]. We consider a general class of boundary or initial value problems for differential equations:

$$\begin{cases} Lu = f(x), & a \le x \le b \\ Bu = 0, \end{cases} \tag{1.1}$$

where L is a differential operator of order M of the form

$$Lu(x) = \sum_{k=0}^{M} p_k(x) u^{(k)}(x), \tag{1.2}$$

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and B is typical Dirichlet, Neumann or mixed boundary conditions. Suppose that $\{p_k\}_{k=0,\dots,M}$ and f are continuous in $[a, b]$ and $p_M(x) > 0$ for $x \in [a, b]$. There is no loss of generality in considering only homogeneous conditions in (1.1) because it is always possible to reduce nonhomogeneous problems to the treated cases, by means of suitable transformations.

2. Description of the method

In order to solve problem (1.1), reproducing kernel spaces $W_2^m[a, b]$ with $m = 1, 2, 3, \dots$ are defined in the following, for more details and proofs we refer to [10].

Definition 2.1. The inner product space $W_2^m[a, b]$ is defined as $W_2^m[a, b] = \{u(x)|u^{(m-1)}$ is absolutely continuous real valued function, $u^{(m)} \in L^2[a, b], Bu = 0\}$. The inner product in $W_2^m[a, b]$ is given by

$$(u(.), v(.))_{W_2^m} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx, \tag{2.3}$$

and the norm $\|u\|_{W_2^m}$ is denoted by $\|u\|_{W_2^m} = \sqrt{(u, u)_{W_2^m}}$, where $u, v \in W_2^m[a, b]$.

According to [1–4,8–10], we have the following theorems and lemmas:

Theorem 2.1. *The space $W_2^m[a, b]$ is a reproducing kernel space. That is, for any $u(.) \in W_2^m[a, b]$ and each fixed $x \in [a, b]$, there exists $K(x, .) \in W_2^m[a, b]$, such that $(u(.), K(x, .))_{W_2^m} = u(x)$. The reproducing kernel $K(x, .)$ can be denoted by*

$$K(x, y) = \begin{cases} \sum_{i=1}^{2m} c_i(y)x^{i-1} & x \le y, \\ \sum_{i=1}^{2m} d_i(y)x^{i-1} & x > y. \end{cases} \tag{2.4}$$

Lemma 2.1. *Let $K(x, y)$ be the reproducing kernel of the space $W_2^m[a, b]$, then*

$$\frac{\partial^{i+j}K(x, y)}{\partial x^i \partial y^j} \in L^2[a, b], \quad 0 \le i + j \le 2m - 1, \tag{2.5}$$

with respect to x or y and

$$\frac{\partial^{i+j}K(x, y)}{\partial x^i \partial y^j}, \quad 0 \le i + j \le 2m - 2, \tag{2.6}$$

is absolutely continuous function in $[a, b]$ with respect to x or y.

Lemma 2.2. *If $u \in W_2^m[a, b]$, then there exists a constant d such that for any $x \in [a, b]$,*

$$|u^{(k)}(x)| \le d\|u\|_{W_2^m}, \quad 0 \le k \le m - 1. \tag{2.7}$$

For more details about reproducing kernel Hilbert spaces $W_2^m[a, b]$ with $m = 1, 2, 3, \dots$ and the method of obtaining their reproducing kernels, see [1–4,10] and references therein. In (1.1), for $m > M$, let $L : W_2^m[a, b] \to W_2^1[a, b]$ and assume L has a bounded inverse $L^{-1} : W_2^1[a, b] \to W_2^m[a, b]$. Define the norm of L^{-1} as follows:

$$\|L^{-1}\| = \sup_{0 \ne v \in W_2^1[a,b]} \frac{\|L^{-1}v\|_{W_2^m}}{\|v\|_{W_2^1}}. \tag{2.8}$$

To see the sufficient conditions for the existence of bounded inverse operator please refer to [11]. For any fixed $x_i \in [a, b]$, let $\varphi_i(.) = r(x_i, .)$, where $r(x, .)$ is reproducing kernel of $W_2^1[a, b]$. Further assume that $\psi_i(.) = (L^*\varphi_i)(.)$, where $L^* : W_2^1[a, b] \to W_2^m[a, b]$ is the adjoint operator of L.

Theorem 2.2. *Let $\{x_i\}_{i=1}^{\infty}$ be dense on $[a, b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^m[a, b]$ and*

$$\psi_i(x) = L_y K(x, y)|_{y=x_i},$$

where the subscript y of operator L_y indicates that the operator L applies to functions of y.