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New nonconforming finite elements on arbitrary convex quadrilateral meshes

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ABSTRACT

In this paper, we construct new nonconforming finite elements on the meshes consisting of arbitrary convex quadrilaterals, especially for the quadratic and cubic cases. For each case, we first define a quadrilateral element that adopts edge moments as the degrees of freedom (DoFs), and then enforce a linear constraint on this element. We have, for the quadratic case, eight degrees of freedom per element and, for the cubic case, eleven DoFs per element, respectively. The dimensions and the bases of different types for the global finite element spaces are provided. We consider the approximations of two-dimensional second order elliptic problems for both of these elements. Error estimates with optimal convergence order in both broken H^1 norm and L^2 norm are given. Moreover, we consider the discretization of the Stokes equations adopting our quadratic element to approximate each component of the velocity, along with piecewise discontinuous P_1 element for the pressure. This mixed scheme is stable and optimal error estimates both for the velocity and the pressure are also achieved. Numerical examples verify our theoretical analysis.

1. Introduction

In the past few years, nonconforming finite elements have gained more and more attractions because of their ability to provide stable solutions of many practical problems in fluid flow and solid mechanics. As an example, for triangular meshes, the simplest nonconforming P_1 element due to Crouzeix and Raviart, employing the degrees of freedom (DoFs) associated with the values at the midpoint of each edge or integral values on each edge, was devised to solve Stokes equations in a stable manner and elasticity problems without numerical locking [1,2]. The key idea, that the integral of the jump of adjacent shape functions along each interface vanishes, is used to pass the patch test [3]. There are also several higher order nonconforming simplicial finite elements listed in literature, see [1,4–7].

Compared with simplicial meshes, quadrilateral meshes usually require less DoFs in practical applications. Many softwares for solving fluid mechanics are available over quadrilateral meshes. Several quadrilateral nonconforming finite elements of first order have been proposed such as the Han element [8], the nonconforming rotated Q_1 (NR) element [9], the DSSY element [10,11] and the enriched NR element [12]. All these elements are of at least four DoFs. It is interesting to see that there exist three-DoF quadrilateral finite elements. For instance, Park and Sheen designed the nonconforming quadrilateral P_1 element in [13], whose DoFs are, as selected in the triangular Crouzeix–Raviart element, midpoint values of the edges of a quadrilateral. Note that the Park–Sheen element utilizes a linear relationship associated with these DoFs for a linear function over a quadrilateral.

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For a higher order quadrilateral nonconforming finite element, the patch test implies that a P_r element needs to satisfy that on each interface the jump of adjacent polynomials be orthogonal to P_{r-1} polynomial on the interface. This hints a simple way to pass the patch test: (i) select the DoFs as values at the r Gauss points on each edge, and then (ii) seek a shape function space containing P_r, such that the restrictions of the shape functions on each edge are polynomials of degree no more than r. Motivated by this consideration, Lee and Sheen constructed an eight-DoF quadratic nonconforming finite element on rectangles [14]. This element is considered as a rectangular version of the triangular Fortin–Soulie element [5,15]. Recently, this idea was extended to the cases $r = 3$ by Meng, Luo and Sheen in [16] and $r > 3$ by Hu and Zhang in [17]. It should be also stressed that the DoFs are not all linearly independent if the shape functions satisfy the condition (ii) as in the case for the Park–Sheen element.

However, it seems quite difficult to design a higher order nonconforming finite element over an arbitrary convex quadrilateral, rather than a rectangle or a parallelogram, adopting the strategy (i)–(ii) if the shape function space completely contains P_r with least DoFs. For the quadratic case, Kim et al. employed spline functions on each quadrilateral to design the piecewise P_2-nonconforming finite element in [18]. As a consequence, its numerical implementation is a little complicated due to the triangulation procedure on each element. Moreover, to the best of our knowledge, there is no such nonconforming finite element of order no less than three that is defined over an arbitrary quadrilateral satisfying the conditions (i) and (ii).

An alternative way to pass the patch test is to employ the edge moments as the DoFs for higher order nonconforming finite elements. A celebrated example is the class of nodal type nonconforming rectangular finite elements of higher order due to Hennart et al. designed in 1988 [19]. Matthies extended these elements for Stokes equations in [20] and pointed out that they are higher order extensions of the NR element. Shortly thereafter, Matthies's elements were modified to be more robust for arbitrary convex quadrilateral meshes [21]. In 2005, Hu and Shi introduced the constrained nonconforming rotated Q_1 (CNR) element in [22] by enforcing a linear constraint involving the edge integrals on the NR element, which interprets the Park–Sheen element from a second perspective.

It is a hot topic to design robust finite elements insensitive to mesh distortion in the field of mathematics [23–26,18, 21,27,9]. Also, numerous techniques have been developed to deal with this issue in the area of engineering. Long and his co-workers introduced the quadrilateral area coordinate method in [28,29]. This method enjoys the advantage that the transformation between the area coordinates and the Cartesian coordinates is always linear, which makes the elements insensitive to mesh distortion. This method was applied to construct robust 4- and 8-node membrane elements [30,31] and a 4-node plate-bending element [32]. The hybrid stress method proposed by Pian et al. [33,34] and the analytic trial function method proposed by Fu et al. [35] are effective techniques as well, which were improved by Cen et al. to develop the 8- and 12-node quadrilateral elements immune to severely distorted meshes even containing concave shapes [36]. Rajendran et al. constructed robust quadrilateral elements by using asymmetric interpolations for the virtual and real displacements in integration of the element stiffness matrix [37,38]. Other successful contributions to this topic include the quasi-conforming method [39], the generalized conforming method [40], the bivariate spline method [41,42] and so on.

In this paper, we construct new higher order nonconforming finite elements on arbitrary convex quadrilaterals, especially for the quadratic and cubic cases. For each case, we select a finite element with the DoFs including the edge moments over a quadrilateral, and then add a linear constraint associated with these moments to accomplish the DoF reduction. The numbers of the DoFs are essentially eight and eleven for quadratic and cubic cases, respectively, due to these linear constraints. The dimensions of the global finite element spaces are counted, and different types of basis functions are provided. On rectangular meshes, our newly proposed quadratic element is precisely the Lee–Sheen element, and our cubic one degenerates to the Meng–Luo–Sheen element [16], provided that some specific shape function spaces are selected. We consider the approximations of second order elliptic problems by using both of these elements. Error estimates with optimal convergence order in both broken H^1 norm and L^2 norm are given. Moreover, we consider the discretization of the Stokes equations using our quadratic element. We adopt this element to approximate each component of the velocity, along with piecewise discontinuous P_1 element for the pressure. This mixed scheme is stable and optimal error estimates both for the velocity and the pressure are also achieved.

The rest of this paper is organized as follows. In Section 2, our new quadratic nonconforming finite element is defined on quadrilateral meshes. The structures of the global finite element spaces are introduced. Section 3 shows our new element of cubic case. Then we apply our elements to the elliptic problems and stationary Stokes equations with optimal error estimate in Section 4. Numerical examples are given in Section 5 to verify our theoretical analysis. Finally we end this paper with some additional remarks in Section 6.

2. A new eight-DoF nonconforming quadratic finite element on quadrilaterals

The main strategy to design our new quadratic nonconforming finite element is to construct a nine-DoF finite element $(K, \mathcal{P}^*_{2,K}, \mathcal{N}^*_{2,K})$ on the quadrilateral K, and then add a linear constraint involving the DoFs of $(K, \mathcal{P}^*_{2,K}, \mathcal{N}^*_{2,K})$ to determine our target element $(K, \mathcal{P}_{2,K}, \mathcal{N}_{2,K})$.

2.1. A nine-DoF nonconforming quadratic finite element

As in Fig. 2.1, let K be an arbitrary convex quadrilateral. The four vertices of K are given by V_1, V_2, V_3, V_4 in a counterclockwise order, and the four edges are denoted as $E_1 = V_1V_2$, $E_2 = V_2V_3$, $E_3 = V_3V_4$ and $E_4 = V_4V_1$, with the