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Boundedness and convergence on fractional order systems

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ABSTRACT

We establish conditions to guarantee boundedness and convergence of signals described by non integer order equations using Caputo derivatives. The case of linear time-varying unforced equations is first studied, and later, results for linear time-varying forced equations and time-varying unforced non linear equations are presented and discussed.

1. Introduction

Though there is no unique nor equivalent definition of non integer derivative, systems defined by Caputo fractional order derivative are widely used because it makes use of initial conditions similar as in the integer order case and also because of the non local behavior, which seems to be the distinctive character that one could expect for non integer dynamics.

Since classical definition of dynamical system (with a specific evolution function, manifold or monoid background and so on) does not completely hold for fractional systems (whether Caputo or other type of derivative is used in its rule of evolution), we will simply call fractional system (of equations) to the object of our study instead of dynamical fractional system.

Like in the integer order case, one of the main topic of research in fractional systems is the study of its asymptotic properties such as convergence and boundedness. In the simplest systems, the linear time invariant systems, those properties can be directly analyzed by using the analytic solution. The reader is referred for example to [1]. The next simplest fractional systems, the linear forced systems and linear time varying systems, which are the main object of study of our work, have received comparatively less attention in the specialized literature. We mention [2] for the latter (scalar case) and [1] for the former (BIBO stability for time invariant systems). Again, in both cases, properties are deduced by appealing to schematic solutions of such equations.

For most complex systems, however, a generic analytic or schematic solution is not possible or not available in the literature and therefore specific tools must be employed or developed instead. Among those tools, we will stand out the Lyapunov functions and the comparison principle [3].

The paper is organized in the following way: Section 2 gives some basic notions and properties of fractional order operators. Section 3 studies fractional linear unforced time variant systems, whereas in Section 4 fractional forced linear systems are analyzed. Next, in Section 5 the study of fractional nonlinear unforced systems is presented. Finally, Section 6 offers general conclusions and future work.

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2. Preliminaries

Some useful definitions and properties (taken mainly from [4] except where indicated) are presented in this section.

Definition 1 (*Fractional Integral [4, page 69]*). The fractional integral of order $\alpha \in \mathbb{R}^+$ of function $f(t)$ on the half axis $\mathbb{R}^+$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{1}$$

where $\Gamma(\alpha) = \int_0^\infty \tau^{\alpha-1} e^{-\tau} d\tau$ is the Gamma function.

We denote $I_T^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_T^t (t-\tau)^{\alpha-1} f(\tau) d\tau$ with $t > T$ and $I_{[T_1,T_2]}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{T_1}^{T_2} (t-\tau)^{\alpha-1} f(\tau) d\tau$ with $T_2 > T_1$.
In the following, $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}$ and $n = [\alpha]$ otherwise, where $[\alpha]$ denotes the integer part of the real number α.

Definition 2 (*Caputo Derivative [5, Definition 3.1]*). The Caputo derivative of order $\alpha \in \mathbb{R}^+$ of function $f(t)$ on the half axis $\mathbb{R}^+$ is defined as

$${}^C D^\alpha f(t) = I^{n-\alpha} f^{(n)}(t) \tag{2}$$

whenever f belongs to $\mathcal{L}^1(a, b)$, the Lebesgue space of functions for which $|f|$ is Lebesgue integrable on the interval (a, b).

It must be noted that Caputo derivative requires that $f^{(n)}$ be differentiable a.e. and if f has well defined Caputo derivative then $f^{(n)}$ must be differentiable a.e.

To simplify the notation, we will denote ${}^C D^\alpha f(t)$ as $D^\alpha f(t)$ or $f^{(\alpha)}$ since we will be using only the Caputo fractional derivative throughout the paper.

An analogue to the fundamental theorem of integer calculus is stated in the next two properties for Caputo fractional derivative.

Property 1 (*[4, Lemma 2.22]*). *If f belongs to $C^n[a, b]$, the space of continuous functions on $[a, b]$ that have continuous first n derivatives (or f belongs to $AC^n[a, b]$, the space of absolutely continuous functions on $[a, b]$ that have continuous first n derivatives), and $\alpha > 0$, then for all $t \in [a, b]$*

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k. \tag{3}$$

Property 2 (*[4, Lemma 2.21]*). *If f belongs to $\mathcal{L}^\infty(a, b)$, the Lebesgue space of bounded functions on the interval (a, b) (or f belongs to $C[a, b]$), and $\alpha > 0$ with $\alpha \notin \mathbb{N}$, then for all $t \in (a, b)$*

$$D^\alpha I^\alpha f(t) = f(t). \tag{4}$$

The next properties will be regularly cited along the proofs of the next sections. It is assumed that $0 < \alpha \le 1$.

Property 3 (*Caputo Derivative Property [6, Lemma 1]*). *Let $x(t) \in \mathbb{R}^n$ be a differentiable vector function, then for all $t \ge 0$ it holds that*

$$D^\alpha x^T x \le 2x^T D^\alpha x. \tag{5}$$

For the proof, the reader is referred to [6]. The proofs of the following two properties can be found at [7].

Property 4 (*Decaying Property*). *If $f(t) \in \mathbb{R}$ is a bounded function that vanishes for all $t > T$ then $I^\alpha f \to 0$ and $D^\alpha f \to 0$ as $t \to \infty$. Moreover, $I^\alpha f$ will be a uniformly continuous function and if $D^\alpha f$ is continuous, $D^\alpha f$ will be a uniformly continuous function.*

Property 5 (*Limit of Integrals Property*). *Let $f(t) \in \mathbb{R}$ be a bounded function, if $I^\alpha f \to L$ as $t \to \infty$ then $I_T^\alpha f \to L$ as $t \to \infty$.*

Finally, we recall the following lemma from [3]

Lemma 1 (*Comparison Principle [3, Lemma 10]*). *If $x(0) = y(0)$ and $D^\beta x \ge D^\beta y$ for $0 < \beta < 1$, then $x(t) \ge y(t)$ for all $t \ge 0$.*