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A modified spectral method for solving operator equations

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ABSTRACT

In this paper we introduce a modified spectral method for solving the linear operator equation

$$Lu = f, \quad L : D(L) \subseteq H_1 \to H_2,$$

where H_1 and H_2 are normed vector spaces with norms $\|.\|_1$ and $\|.\|$, respectively and $D(L)$ is the domain of L. Also for each $h \in H_2$, $\|h\|^2 = (h, h)$ where $(., .)$ is an inner product on H_2. In this method we make a new set $\{\psi_n\}_{n=0}^{\infty}$ for H_1 using L and two sets in H_1 and H_2. Then using the new set $\{\psi_n\}_{n=0}^{\infty}$ we solve this linear operator equation. We show that this method does not have some shortcomings of spectral method, also we prove the stability and convergence of the new method. After introducing the method we give some conditions that under them the nonlinear operator equation $Lu + Nu = f$ can be solved. Some examples are considered to show the efficiency of method.

1. Introduction

There are various methods for solving operator equation

$$Lu + Nu = f,$$

such as spectral [1–5], finite element [6–8], meshless [9–11] and finite difference methods [12,13]. Some method developed for solving the operator equations are analytical or semi-analytical such as Adomian decomposition method [14–20], homotopy analysis and homotopy perturbation methods [21–30] and variational iteration method [31–38]. The spectral method is a famous method so we would like to modify the spectral method such that the modified one hasn't some shortcomings of spectral method. First we consider the linear operator equation

$$Lu = f, \quad L : D(L) \subseteq H_1 \to H_2, \tag{1}$$

where H_1 is a normed vector space over a field $\mathbb{F}_1$ with norm $\|\cdot\|_1$ and H_2 is an inner product space over a field $\mathbb{F}_2$ with an inner product $(., .)$ and norm

$$\|h\|^2 = (h, h), \quad h \in H_2.$$

Let $\{\phi_i\}_{i=0}^{\infty}$ be a Schauder basis for H_1 i.e. for each $h \in H_1$ there exists a unique sequence $\{\alpha_i\}_{i=0}^{\infty}$ in the field $\mathbb{F}_1$ such that

$$h = \sum_{i=0}^{\infty} \alpha_i \phi_i. \tag{2}$$

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Assume that the linear problem (1) has a unique solution u. In the spectral method we put

$$u'_n = \sum_{i=0}^{n} \alpha_i^n \phi_i, \tag{3}$$

in the linear operator equation and calculate $\{\alpha_i^n\}_{i=0}^n$ from

$$(Lu'_n - f, \varphi_j) = 0, \quad 0 \le j \le n, \tag{4}$$

where $\{\varphi_i\}_{i=0}^\infty$ is a Schauder basis for H_2. Now we introduce the following shortcomings of spectral method.

1. Clearly (4) forms a linear system that we are not sure the matrix of coefficients is invertible, for example let $L = g(t,x)\frac{\partial}{\partial t} - 2\frac{\partial^2}{\partial x^2}$ and H_1 and H_2 be $L^2([0,1] \times [-1,1])$ with the inner product

$$(g_1, g_2) = \int_{-1}^{1}\int_{0}^{1} g_1(t,x) g_2(t,x) dt dx,$$

and $\Phi_{n,m}(t,x) = P_{n+1}(2t-1)P_m(x),\ 0 \le n \le 3,\ 0 \le m \le 4$ where $\{P_n\}_{n=0}^\infty$ are Legendre polynomials and $g(t,x) = P_5(2t-1)P_5(x)$ then for each $f \in L^2([0,1]\times[-1,1])$ equations

$$\left(L \sum_{i=0}^{3}\sum_{j=0}^{4} \alpha_{i,j}^{3,4} \Phi_{i,j} - f, \Phi_{r,k} \right) = 0, \quad 0 \le r \le 3,\ 0 \le k \le 4,$$

form a linear system such that the matrix of coefficients is not invertible.

2. Assume that we have calculated $\{\alpha_i^n\}_{i=0}^n$ from (4), for calculating $\{\alpha_i^{n+1}\}_{i=0}^{n+1}$ we must solve a new linear system and we cannot deduce that

$$\alpha_i^{n+1} = \alpha_i^n, \quad i = 0, \ldots, n.$$

3. In the spectral method we cannot conclude that $u'_n = P''_n u$ where u is the exact solution of linear problem (1) and P''_n is a bounded projection operator from H_1 onto M'_n such that

$$P''_n \sum_{k=0}^{m} d_k \phi_k = \sum_{k=0}^{n} d_k \phi_k, \quad \forall\, m \in \mathbb{N}_0, \tag{5}$$

$$M'_n = \mathrm{span}\{\phi_0, \ldots, \phi_n\}. \tag{6}$$

In this article we will introduce the modified spectral method. For this aim we will construct a new set $\{\psi_i\}_{i=0}^\infty$ using two predetermined sets $\{\phi_i\}_{i=0}^\infty$ and $\{\varphi_i\}_{i=0}^\infty$ such that solving the linear problem (1) using this new set hasn't the above shortcomings. Also we will prove the stability and convergence of method.

2. An analytical method

In this section we want to introduce an analytical method for solving the linear operator equation

$$Lu = f, \tag{7}$$

where

$$L : D(L) \subseteq H_1 \to H_2, \tag{8}$$

where H_1 is a normed vector space with norm $\|\cdot\|_1$ and H_2 is an inner product space with an inner product $(.,.)$ and norm

$$\|h\|^2 = (h,h), \quad h \in H_2.$$

For this purpose we employ the following sets

$$\{\phi_0, \phi_1, \ldots, \phi_n, \ldots\} \subseteq D(L), \tag{9}$$

$$\{\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots\} \subseteq H_2, \tag{10}$$

then we construct a new set $\{\psi_n\}_{n=0}^\infty$ such that

$$(L\psi_n, \varphi_{N_n}) \ne 0, \qquad (L\psi_n, \varphi_z) = 0, \quad z < N_n,$$

where $N_n \ge n$ and $N_i < N_j$ for $i < j$. Therefore if the second set (10) be an orthogonal basis for $R(L)$ then the new set $\{\psi_n\}_{n=0}^\infty$ has the following property

A. $L\psi_n = \sum_{k=0}^\infty b_{k,n}\varphi_k$, where $b_{k,n} = 0$ for $k < N_n$, $b_{N_n,n} \ne 0$ where $N_n \ge n$ and $N_i < N_j$ for $i < j$.

Constructing the new set $\{\psi_n\}_{n=0}^\infty$ and putting $u_k = \sum_{n=0}^k a_n^k \psi_n$ in problem (7) and calculating $\{a_n^k\}_{n=0}^k$ from

$$(f - Lu_k, \varphi_{N_i}) = 0, \quad i = 0, \ldots, k,$$