Journal of Computational and Applied Mathematics 292 (2016) 150–173

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A duality-based path-following semismooth Newton method for elasto-plastic contact problems

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ARTICLE INFO

Article history:
Received 6 September 2014
Received in revised form 18 April 2015

MSC:
49M15
49M29
74C05
74S05

Keywords:
Elasto-plastic contact
Variational inequality of the 2nd kind
Fenchel duality
Moreau–Yosida/Tikhonov regularization
Path-following
Semismooth Newton

ABSTRACT

A Fenchel dualization scheme for the one-step time-discretized contact problem of quasi-static elasto-plasticity with combined kinematic–isotropic hardening is considered. The associated path is induced by a coupled Moreau–Yosida/Tikhonov regularization of the dual problem. The sequence of solutions to the regularized problems is shown to converge strongly to the optimal displacement–stress–strain triple of the original elasto-plastic contact problem in the space-continuous setting. This property relies on the density of the intersection of certain convex sets which is shown as well. It is also argued that the mappings associated with the resulting problems are Newton—or slantly differentiable. Consequently, each regularized subsystem can be solved mesh-independently at a local superlinear rate of convergence. For efficiency purposes, an inexact path-following approach is proposed and a numerical validation of the theoretical results is given.

1. Introduction

In this paper we consider the quasi-static elasto-plasticity model with an associative flow law (sometimes called Prandtl–Reuss normality law) and von Mises hardening under the small strain assumption set forth in [1]. First investigations of the elasto-plastic problem from a mathematical point of view can be found in [2,3], where [3] includes existence for the fully continuous case. Numerical analysis of the semi-discrete and fully-discrete versions can be found, for example, in [4,1]. Appropriate discretization schemes for plasticity problems with hardening have been investigated extensively in the recent past. Here we only mention [5–8] for adaptive finite element methods. Concerning numerical solution methods, we refer to the multigrid approach in [9], various generalized Newton methods in finite dimensions [10–12,9,13], including the standard return mapping algorithm in [14] as well as interior point strategies, cf. e.g. [15].

A general introduction to elastic contact problems including corresponding numerical approaches can be found in the monographs [16,17], and multigrid methods for elastic contact are analyzed, e.g., in [18–20], where the latter references are devoted to two-body contact. For the treatment of elastic friction problems we refer to [21,20] as well as to the efficient active set algorithm proposed in [22]. Subspace correction methods for variational inequalities of the second kind with application to frictional contact have been investigated in [23]. In [10,24] plastic material behavior is incorporated in addition to the contact constraints. In the latter references the elasto-plastic friction problem is reformulated utilizing a nonlinear complementarity problem (NCP) function yielding a nonsmooth system which can be solved efficiently by applying a generalized Newton method in a discrete framework provided a set of damping parameters is chosen appropriately.

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http://dx.doi.org/10.1016/j.cam.2015.06.010
0377-0427/

While some attention has been paid to infinite-dimensional methods in linear elasticity with (frictional) contact [25,26], elasto-plastic problems are still less researched. Among the few available references we mention [27] for domain decomposition methods leading to a linear rate of convergence.

In this paper, we introduce a path-following semismooth Newton method which admits a rigorous convergence analysis in the continuous setting and each path-problem can be solved at a local superlinear rate and in a mesh-independent way upon discretization. Mesh-independent convergence is often characterized by iteration numbers of the underlying problem solver which are uniformly bounded, or, in the ideal case, essentially constant with respect to the mesh size. In the context of semismooth Newton (SSN) methods, mesh-independence typically refers to the fact that for sufficiently fine meshes the convergence quotients are stable with respect to the mesh size in a neighborhood of the solution [28]. For many numerical approaches to variational inequality problems this desirable property cannot be proven. This may result in a considerable computational overhead when computing on very fine meshes, see for instance [29, Table 5.3] in the case of state-constrained optimal control problems.

With regard to semismooth Newton methods, mesh-independent convergence requires the generalized differentiability of the nonlinear mapping associated with the root finding problem. For constrained minimization problems, this property is closely related to sufficiently high regularity of the solution (or the Lagrange multipliers). Such an increased regularity is often not available. For instance, the SSN approach to plasticity problems without contact constraints in [11] turns out to be problematic as far as function space convergence is concerned. In fact, due to the lack of a sufficient norm gap between domain and image space of the mapping involved in the underlying nonsmooth system in the displacement variable, generalized differentiability in the sense of [30] does not hold true. The resulting lack of a well-defined infinite-dimensional generalized Newton iteration usually results in a mesh-dependent solver.

The outline of the paper is as follows. In Section 2 we formulate the elasto-plastic contact problem in the mathematical language of the monograph [31]. Section 3 contains a specific reformulation suitable for application of the Fenchel duality theory from convex analysis. It is used to derive an equivalent dual problem in terms of two stress-related variables. As a variational inequality problem of the first kind, the dual problem turns out to be structurally simpler as the original problem which is a variational inequality of the mixed (i.e. first and second) kind. In the subsequent section a suitable regularization for the dual problem is introduced which is proven to converge to the original problem under certain density requirements on the regularization space (Appendix A). In this regard Appendix B is dedicated to prove that these density properties are indeed fulfilled in all relevant cases. In contrast to the original problem, the regularized problems from Section 4 can be solved by the SSN method in infinite dimensions and the convergence of the resulting solver is studied, cf. Section 5. It should be emphasized that the entire convergence analysis is valid in, both, the two- and three-dimensional case. In the last part the infinite-dimensional setting is left and a simple conforming finite element discretization is proposed. We further derive the discrete version of the solver and the original problem is approximated by a path-following strategy with respect to the regularization parameters. To verify the theoretical properties of the solver, we present numerical results for two elasto-plastic contact problems in 2D which support the theoretical results of this work.

2. Problem formulation

The starting point of our analysis is the small-strain elasto-plastic contact problem in the displacement u, the plastic strain p and a set of internal variables ξ which model the evolution of a body subject to given applied forces. The body is represented by a bounded domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, with $N^{0,1}$-property [32] and it adheres to a fixed part $\Gamma_d \subset \partial\Omega$ with positive surface measure. We further denote by $\Gamma_n \subset \partial\Omega \setminus \Gamma_d$ some relatively open part of the boundary where a given surface load $g \in L^2(\Gamma_n)$ is applied. A given volume force density is denoted by $f \in L^2(\Omega)$. The elasto-plastic behavior at a material point $x \in \Omega$ is determined by a given yield criterion leading to a dissipation functional which typically is nonsmooth, lower semicontinuous (l.s.c.) and convex [1]. Often, the displacement of the body is restricted by a given rigid obstacle giving rise to an elasto-plastic contact problem. Therefore we fix a set $\Gamma_c \subset \partial\Omega$ which potentially contains the contact region with the obstacle. We emphasize here that the approach presented in this work does not hinge on $\Gamma_c \neq \emptyset$. To measure the gap between Ω and the obstacle we use a given function

$$\psi \in Z := H^{1/2}(\Gamma_c) \quad \text{with } \psi \geq 0 \text{ almost everywhere (a.e.) on } \Gamma_c;$$

see [17]. For the time being we neglect frictional forces such that in terms of the variational formulation, we incorporate the contact constraint by a kinematic non-penetration condition on the displacement u:

$$\tau_n u \leq \psi \quad \text{on } \Gamma_c, \tag{2.1}$$

where $\tau_n : [H^1_{0,\Gamma_d}(\Omega)]^N \to Z, u \mapsto (\tau|_{\Gamma_c}(u)) \cdot n$ denotes the normal trace mapping restricted to Γ_c. For analytical reasons we assume that Γ_c is relatively open with $N^{1,1}$-property and C^∞-boundary $\partial\Gamma_c$. For simplicity and without loss of generality we further stipulate

$$\overline{\Gamma}_c \subset \Sigma, \tag{2.2}$$

where Σ denotes the interior of $\partial\Omega \setminus \Gamma_d$ in $\partial\Omega$, to avoid working with the space $H^{1/2}_{00}(\Gamma_c)$. Concerning the splitting of the boundary we further assume

$$\partial\Omega = \overline{\Gamma}_c \cup \overline{\Gamma}_n \cup \overline{\Gamma}_d, \qquad \Gamma_c \cap \Gamma_n \cap \Gamma_d = \emptyset, \quad \partial\Sigma \in C^\infty.$$