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A posteriori error estimates of discontinuous Galerkin methods for the Signorini problem

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ABSTRACT

A reliable and efficient a posteriori error estimator is derived for a class of discontinuous Galerkin (DG) methods for the Signorini problem. A common property shared by many DG methods leads to a unified error analysis with the help of a constraint preserving enriching map. The error estimator of DG methods is comparable with the error estimator of the conforming methods. Numerical experiments illustrate the performance of the error estimator.

1. Introduction

Adaptive finite element methods based on reliable and efficient *a posteriori* error estimators are playing an important role in the numerical methods for solving partial differential problems. The monographs [1,2] present an introductory as well as comprehensive study of the subject. In this article, we study the derivation of a reliable and efficient *a posteriori* error estimator for a class of discontinuous Galerkin (DG) methods for the Signorini problem. The Signorini problem is a variational inequality of the first kind that arises from the study of frictionless contact problems [3]. Recently in [4], several DG methods have been proposed and their *a priori* error analysis has been derived for the Signorini problem. Independently in [5], a local discontinuous Galerkin (LDG) method has been proposed and analyzed for a simplified Signorini problem. There are handful works on the analysis of finite element methods for variational inequalities of the first kind. The work related to the obstacle problem can be found in [3,6–11] and in [12–17]. Related to the work on the Signorini problem, we refer to [3,18–21] for *a priori* error analysis and to [22–25] for *a posteriori* analysis. The analysis in [24,25] is for a mixed formulation of the Signorini problem introducing a Lagrange multiplier and the analysis in [22,23] is for a simplified Signorini problem. Also refer to the recent work in [26] for the a posteriori error control of conforming methods. In this article, we derive a residual based *a posteriori* error estimator for a class of DG methods for the Signorini problem. In the subsequent analysis of DG methods, a constraint preserving enriching map connecting DG functions with conforming finite element functions plays an important role. Also a common property shared by many DG methods is helpful in deriving the analysis in a unified framework. The difficulties in the analysis stem from the variational inequality and the nonconformity of the finite element space. We treat them by introducing a Lagrange multiplier and a nonlinear enriching function.

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The rest of the article is organized as follows. In Section 2, we introduce the Signorini problem and derive some useful properties pertaining to the solution and a Lagrange multiplier corresponding to constraints on the contact region. In Section 3, we introduce the notation and recall some preliminary results. Therein, we construct an enriching map preserving constraints on the discrete functions. We introduce the discrete problem in Section 4 and derive *a posteriori* estimates in Section 5. In Section 6, we discuss some applications of the analysis to DG methods. In Section 7, we present some numerical experiments to illustrate the theoretical results. Finally, we present some conclusions in Section 8.

2. Model problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega = \Gamma$. We assume that the boundary Γ is partitioned into three parts Γ_D, Γ_N and Γ_C which are open and mutually disjoint sets with $\text{meas}(\Gamma_D) > 0$. Further, assume that $\bar{\Gamma}_C \subset \partial\Omega \backslash \bar{\Gamma}_D$ (see [19, p. 112]) and the unit outward normal vector ν to Γ_C is constant [25].

The linearized strain tensor ε and stress tensor σ are defined, respectively, by

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right), \tag{2.1}$$

$$\sigma(\mathbf{u}) = \lambda\,(\text{tr}\varepsilon(\mathbf{u}))I + 2\mu\,\varepsilon(\mathbf{u}), \tag{2.2}$$

where $\mathbf{u} : \Omega \to \mathbb{R}^2$ is the displacement vector, $\lambda > 0$ and $\mu > 0$ are the Lame's coefficients and I denotes 2×2 identity matrix. Define the space $\mathbf{V}$ by

$$\mathbf{V} = [H^1_D(\Omega)]^2 := \{\mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}, \tag{2.3}$$

where $H^1(\Omega)$ is the standard Sobolev space equipped with the norm

$$\|v\|_{1,\Omega} = \left(\sum_{0\le|\alpha|\le 1} \|\partial^\alpha v\|^2_{L_2(\Omega)}\right)^{1/2}.$$

Here $\alpha = (\alpha_1, \alpha_2)$ is a multi-index with α_i's ($i = 1, 2$) are nonnegative integers and $|\alpha| = \alpha_1 + \alpha_2$. For simplicity, the $L_2(\Omega)$ norm will be denoted by $\|\cdot\|$.

For a vector valued function $\mathbf{v}$, define its normal and tangential components on the boundary by $v_n = \mathbf{v}\cdot\mathbf{n}$ and $\mathbf{v}_t = \mathbf{v} - v_n\mathbf{n}$, respectively, where $\mathbf{n}$ is the unit outward normal vector and $\mathbf{t}$ is the tangential vector to $\partial\Omega$. In a similar way, for any matrix valued function $\mathbf{q}$, its normal and tangential components are defined by $q_n = (\mathbf{qn})\cdot\mathbf{n}$ and $\mathbf{q}_t = \mathbf{qn} - q_n\mathbf{n}$, respectively.

A nonempty, closed and convex subset $\mathcal{K}$ of $\mathbf{V}$ is defined by

$$\mathcal{K} := \{\mathbf{v} \in \mathbf{V} : v_n \le 0 \text{ a.e. on } \Gamma_C\}. \tag{2.4}$$

Given $\mathbf{f} \in [L_2(\Omega)]^2$ and $\mathbf{g} \in [L_2(\Gamma_N)]^2$, the variational formulation for the Signorini problem is to find $\mathbf{u} \in \mathcal{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \ge l(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathcal{K}, \tag{2.5}$$

where the bilinear form $a(\cdot,\cdot)$ and the linear functional l are defined by

$$a(\mathbf{w}, \mathbf{v}) = \int_\Omega \sigma(\mathbf{w}) : \varepsilon(\mathbf{v})\,dx \quad \forall\, \mathbf{v}, \mathbf{w} \in \mathbf{V}, \tag{2.6}$$

$$l(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx + \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\,ds \quad \forall\, \mathbf{v} \in \mathbf{V}, \tag{2.7}$$

and for any $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{2\times 2}$, $\mathbf{B} = (b_{ij}) \in \mathbb{R}^{2\times 2}$, ":" is defined as

$$\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij}b_{ij}.$$

It is easy to check that the bilinear form $a(\cdot,\cdot)$ is $\mathbf{V}$-elliptic and continuous. From the theory of elliptic variational inequalities [3,18,27], it is well known that the problem (2.5) admits a unique solution.

In the subsequent analysis, we will be dealing with the traces of Sobolev functions. We recall the following results from the trace theory (see [19,28]):

The assumptions on Γ_C imply that [19, p. 88] the trace operator $\gamma_c : \mathbf{V} \to [H^{1/2}(\Gamma_C)]^2$ maps $\mathbf{V}$ continuously onto $[H^{1/2}(\Gamma_C)]^2$, where the space $[H^{1/2}(\Gamma_C)]^2$ is equipped with the norm

$$\|\mathbf{w}\|_{1/2,\Gamma_C} = \left(\|\mathbf{w}\|^2_{L_2(\Gamma_C)} + \int_{\Gamma_C}\int_{\Gamma_C} \frac{|\mathbf{w}(x) - \mathbf{w}(y)|^2}{|x-y|^2}\,dx\,dy\right)^{1/2}. \tag{2.8}$$