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Minimizing control variation in discrete-time optimal control problems

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HIGHLIGHTS

- We present a gradient-based method for discrete-time optimal control problems.
- The cost function is the sum of terminal cost and the variation of control signal.
- A smooth transformation and the constraint transcription technique are used.
- Two examples are provided to demonstrate the feasibility of the method.

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ABSTRACT

For a real practical system, a large fluctuation in the control signal is highly undesirable. To address this undesirable situation, we investigate a discrete-time optimal control problem subject to terminal state and all-time-step constraints on the state and control, where the cost function is the sum of terminal cost and the variation of the control signal. The variation of the control signal is expressed in terms of absolute value functions and hence is non-smooth. By a novel smooth transformation and the constraint transcription technique, this problem is approximated by a constrained discrete-time optimal control with the new cost function involves only smooth functions. A gradient-based computational method is then derived, which is supported by rigorous convergence analysis. Two examples are provided to demonstrate the effectiveness and advantages of the proposed method.

1. Introduction

It is well known that in a standard optimal control problem, the cost function is usually the sum of the terminal cost and the integral cost [1,2]. The cost on changing control is a cost which is associated with changing control from one discrete value to another. It is first proposed for a continuous time optimal control problem in [3], and later in [4–6]. This type of cost is of practical importance, because in many real systems, a change of the control action will always induce a cost. The cost could be as trivial as wear and tear on the system's actuators or as complex as loss of confidence in national economy.

In [3], necessary conditions for optimality are derived for an optimal control problem, in which the control signal can assume two possible values and there is a cost associated with changing the control signal from one value to another. In [5],

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optimality conditions are obtained for a more general optimal control problem in which the total variation of the control signal is incorporated as a penalty to the cost function, forming a new augmented cost function. However, these methods do not consider state constraints which are always encountered in real-world applications. Subsequently, the control parametrization method, which is implemented in the MISER optimal control software [7], is introduced in [8] to derive a gradient-based computational method for solving a general class of continuous time optimal control problems where the total variation of the control signal is included in the cost function, while the state variables are subject to inequality path constraints. In [4], a superior gradient-based computational approach is proposed, where the path constraints are allowed to contain both state and control variables.

Discrete-time optimal control problems arise in many multistage control and scheduling problems [6]. However, it appears that discrete-time optimal control problems with cost on the changes of the control action have not yet been addressed in the literature.

Numerous methods are now available for solving discrete-time optimal control problems. They can be categorized into four main classes: (1) methods based on the maximum principle [9,10]; (2) methods using mathematical programming techniques [11,12]; (3) methods based on a hybrid of maximum principle and mathematical programming techniques such as those reported in [13,14]; and (4) methods using the dynamic-programming technique [15,16].

In [17], a gradient-based computational method is developed for a general class of discrete-time optimal control problems subject to all-time-step constraints on the state and control variables, where the all-time-step constraints are approximated by an inequality constraint in summation form through the applications of the constraint transcription technique [18].

In the existing literature, the cost function of a discrete-time optimal control problem is usually a function of the final state reached by the system and/or a summation term involving the state and control variables at each time-step. The cost of changing the control signal is totally ignored.

In this paper, we consider a general discrete-time optimal control problem subject to the terminal state and the all-time-step constraints on the state variables, where the cost function is the sum of terminal cost and the variation of the control signal. We will derive an effective gradient-based computational method for solving this class of constrained discrete-time optimal control problems.

Our method involves two stages. The first stage is the equivalence transformation of the cost function. We first note that the cost function is the sum of the terminal cost and the variation of the control. Note that, the variation of the control is expressed as the sum of absolute value functions, where each of which is non-smooth. The novel transformation procedure introduced in [4] is applied to transform the non-smooth absolute value function into an equivalent smooth function subject to smooth constraints. The second stage is the application of the constraint transcription introduced in [18] to approximate the all-time-step constraints by an inequality in summation form. The gradient formulae for the cost and constraint functions and their calculations are given in Section 5. With these gradient formulae, each of the approximate problems can be solved effectively by any existing constrained optimization technique. For example, the software package, DMISER [7], for the discrete-time optimal control problems can be used. The convergence properties are given in Section 4 to support the proposed numerical method. Two examples are computed in Section 6 to demonstrate the feasibility and versatility of the method.

2. Problem formulation

Consider a discrete-time dynamic system described by following system of difference equations:

$$x(k+1) = f(k, x(k), u(k)), \quad k = 0, 1, \dots, M-1, \; x(0) = x^0, \tag{1}$$

where

$$x = [x_1, \dots, x_n]^\top \in \mathbb{R}^n, \qquad u = [u_1, \dots, u_r]^\top \in \mathbb{R}^r$$

are, respectively, state and control vectors; $f = [f_1, \dots, f_n]^\top : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is a given function; $x^0 = [x_1^0, \dots, x_n^0]^\top$ is the initial state vector.

Define

$$U = \{\nu = [v_1, \dots, v_r]^\top \in \mathbb{R}^r : \alpha_i \le v_i \le \beta_i, i = 1, \dots, r\}, \tag{2}$$

where $\alpha_i,\ i = 1, \dots, r$, and $\beta_i,\ i = 1, \dots, r$, are given real numbers. Note that U is a compact and convex subset of $\mathbb{R}^r$.

Let u denote a control sequence $\{u(k),\ k = 0, 1, \dots, M-1\}$ in U, which is written as:

$$u = [u(0)^\top, u(1)^\top, \dots, u(M-1)^\top]^\top. \tag{3}$$

Here, u is called an **admissible control**. Let $\mathcal{U}$ be the class of all such admissible controls.

For each $u \in \mathcal{U}$, let $x(k|u),\ k = 0, 1, \dots, M$, be a sequence in $\mathbb{R}^n$ such that the system (1) is satisfied. This discrete-time function is called the **solution** of system (1) corresponding to the control $u \in \mathcal{U}$.

Let $u_i(k)$ denote the ith component of the control signal at time k. Then the **variation** of u_i is defined by

$$\bigvee_0^{M-1} u_i = \sum_{k=0}^{M-2} |u_i(k+1) - u_i(k)|.$$