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# A finite element method for singular solutions of the Navier–Stokes equations on a non-convex polygon

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ABSTRACT

It is shown in Choi and Kweon (2013) that a solution of the Navier–Stokes equations with no-slip boundary condition on a non-convex polygon can be written as $[\mathbf{u}, p] = \mathcal{C}_1[\Phi_1, \phi_1] + \mathcal{C}_2[\Phi_2, \phi_2] + [\mathbf{u}_R, p_R]$ near each non-convex vertex, where $[\mathbf{u}_R, p_R] \in \mathbf{H}^2 \times \mathrm{H}^1$, $[\Phi_i, \phi_i]$ are corner singularity functions for the Stokes problem with no-slip condition, and $\mathcal{C}_i \in \mathbb{R}$ are coefficients which are called the stress intensity factors. We design a finite element method to approximate the coefficients $\mathcal{C}_i$ and the regular part $[\mathbf{u}_R, p_R]$, show the unique existence of the approximations, and derive their error estimates. Some numerical examples are given, confirming convergence rates for the approximations.



## 1. Introduction

Our concern is with a stable finite element scheme handling the singular behavior at the corner of solutions for the stationary Navier–Stokes equations on a (non-convex) polygon. For this we employ a known decomposition (see [1]) of the solution into a singular part plus a regular part near the non-convex vertex. The singular part consists of the multiplication of the coefficients and the leading corner singularities for the Stokes operator with no-slip boundary condition on the infinite sector. It is seen from [1] that each coefficient of the corner singularities is regarded as a continuous linear functional on the space $\mathbf{H}^{s-2} \times \mathrm{H}^{s-1}$ for $s \tilde{\gamma} \lambda_i + 1$, called the stress intensity factor, and the regular part is known to have the $\mathbf{H}^2 \times \mathrm{H}^1$-regularity provided that the given force has the $\mathbf{L}^2$-regularity, where $\lambda_i \in (1/2, 1)$ denotes the singular exponent. In this paper we propose a finite element scheme for the stress intensity factors and the regular part.

Let $\Omega$ be a non-convex polygon in the plane $\mathbb{R}^2$ with the boundary $\Gamma = \partial\Omega$. The stationary Navier–Stokes equations to be considered in this paper are

$$
\begin{aligned}
&-\mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,\\
&\mathrm{div}\mathbf{u} = g \quad \text{in } \Omega, \\
&\mathbf{u} = 0 \quad \text{on } \Gamma,
\end{aligned} \tag{1.1}
$$

where $\mathbf{u}$ is the velocity vector, $p$ is the pressure; $\mu$ is a viscous number with $\mu\tilde{\gamma}0$; $\mathbf{f}$ and $g$ are given functions with $\int_\Omega g\,d\mathbf{x} = 0$. Usually the right hand side $g$ is set to be zero for incompressibility but the non-vanishing $g \neq 0$ is for specifying the regularity of $g$ in the coefficients of the corner singularities (see (2.1)). In this paper we assume that the boundary $\Gamma$

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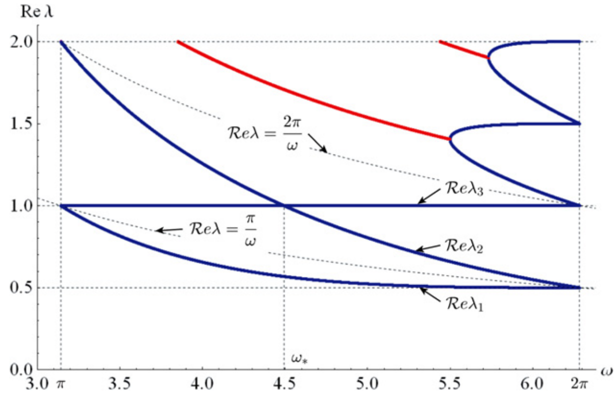


**Fig. 1.** The singular exponents $\lambda_i$.

has only one non-convex vertex $P$, for simplicity. Denote by $\omega := \omega_2 - \omega_1 \tilde{\gamma} \pi$ an opening angle at $P$, where $\omega_i$ are numbers satisfying $\omega_1 < \omega_2 < \omega_1 + 2\pi$.

Throughout this paper we will use the corner singularity functions for the Stokes operator $\mathbf{L}[\mathbf{v}, q] = [-\mu\Delta\mathbf{v}+\nabla q, -\mathrm{div}\mathbf{v}]$ with no-slip boundary condition (see [2, Section 5.1]). The leading singular eigenvalues $\lambda_i$, which are the roots of the trigonometric equation: $\sin^2(\lambda\omega) - \lambda^2 \sin^2 \omega = 0$, are real and ordered by

$$\text{Case 1: } 1/2 < \lambda_1 < \pi/\omega < \lambda_2 = 1 < \lambda_3 < 2\pi/\omega, \quad \omega \in (\pi, \omega_*], \tag{1.2a}$$

$$\text{Case 2: } 1/2 < \lambda_1 < \pi/\omega < \lambda_2 < \lambda_3 = 1 < 2\pi/\omega, \quad \omega \in (\omega_*, 2\pi), \tag{1.2b}$$

where $\omega_* \approx 1.4303\pi$ is the solution of the equation: $\tan\omega = \omega$ in the interval $(0, 2\pi]$. Throughout this paper we shall consider Case 2 for a generalization. Letting the non-convex vertex be placed at the origin, the corner singularities for velocity and pressure are given by

$$\Phi_i = \mu^{-1}\chi r^{\lambda_i}\mathcal{T}_i(\theta), \qquad \phi_i = \chi r^{\lambda_i - 1}\xi_i(\theta), \tag{1.3}$$

where $\mathcal{T}_i(\theta)$ and $\xi_i(\theta)$ are certain smooth trigonometric eigenfunctions for velocity and pressure, corresponding to the eigenvalue $\lambda_i$, and $\chi$ is a smooth cutoff function which has a support near the origin. Obviously the function $\mathcal{T}_i$ satisfies $\mathcal{T}_i(\omega_1) = \mathcal{T}_i(\omega_2) = 0$. In particular, for $\lambda_i = 1$, the singularities are $\Phi_i = 0$ and $\phi_i = 4\chi$, which are smooth functions. See Fig. 1.

In what follows, for an open set $D \subset \mathbb{R}^2$, $W^{s,q}(D)$ denotes the usual fractional Sobolev space with norm $\|v\|_{s,q,D}$ and $L^\infty(D)$ the measurable space with norm $|v|_{\infty,D} := \mathrm{ess\,sup}\{|v(\mathbf{x})| : \mathbf{x} \in D\}$ (see [3]). If $q = 2$, denote by $H^s(D) = W^{s,2}(D)$ and $\|v\|_{s,D} = \|v\|_{s,2,D}$. Also we define $H^1_0(D) = \{v \in H^1(D) : v|_{\partial D} = 0\}$, $L^2_0(D) = \{v \in L^2(D) : \int_D v\,d\mathbf{x} = 0\}$, $H^s_0(D) = H^s(D) \cap H^1_0(D)$ for $s \geq 1$, and $\bar{H}^s(D) = H^s(D) \cap L^2_0(D)$ for $s \geq 0$. Let $H^{-s}(D)$ be the dual space of $H^s_0(D)$ with $\|f\|_{-s,D} := \sup\{\langle f, v\rangle/\|v\|_{s,D} : 0 \neq v \in H^s_0(D)\}$, where $\langle\,,\,\rangle$ means the duality pairing. Similarly $\bar{H}^{-s}(D)$ denotes the dual space of $\bar{H}^s(D)$ and $H'$ also means the dual space of H. Furthermore denote by $\mathbf{H}^s(D) = H^s(D) \times H^s(D)$ and similar for other spaces. If $D = \Omega$, we will omit the domain $\Omega$ in spaces and norms such as $H^s = H^s(\Omega)$ and $\|v\|_s = \|v\|_{s,\Omega}$. Throughout this paper let $C\tilde{\gamma}0$ be a generic constant only depending on the domain $\Omega$. Define the number $s_i = \lambda_i + 1$, which is used in classifying the regularity order space of the solution depending on the corner singularity.

We state a basic corner singularity result for the Navier–Stokes equations (1.1).

**Theorem 1.1** (*Theorem 1.2 in [1]*). *Suppose that $\mathbf{f} \in \mathbf{H}^{-1}$ and $g \in L^2_0$. Let*

$$\mathcal{N} = \sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\tilde{\mathbf{V}}} \frac{\int_\Omega [(\mathbf{w}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}d\mathbf{x}}{\|\nabla\mathbf{u}\|_0\|\nabla\mathbf{v}\|_0\|\nabla\mathbf{w}\|_0},$$

*where $\tilde{\mathbf{V}} = \{\mathbf{v} \in \mathbf{H}^1_0 : \mathrm{div}\,\mathbf{v} = 0\}$. If $\mu\tilde{\gamma}[C_1\mathcal{N}(\|\mathbf{f}\|_{-1} + \mu\|g\|_0)]^{1/2} + C_1\|g\|_0$ for some given constant $C_1\tilde{\gamma}0$, there is a unique solution $[\mathbf{u}, p] \in \mathbf{H}^1_0 \times L^2_0$ of the problem (1.1), with $\mu\|\mathbf{u}\|_1 + \|p\|_0 \leq C_a(\|\mathbf{f}\|_{-1} + \mu\|g\|_0)$ for some constant $C_a$.*

*Furthermore, for a number s satisfying $\pi/\omega + 1 < s_2 < s \leq 2$ we assume that $\mathbf{f} \in \mathbf{H}^{s-2}$ and $g \in \bar{H}^{s-1}$. If $\mu\tilde{\gamma}C_b(\|\mathbf{f}\|_{s-2} + \mu\|g\|_{s-1})^{1/2}$ for some given constant $C_b\tilde{\gamma}0$, there are some coefficients $\mathcal{C}_i$ and a regular part $[\mathbf{u}_R, p_R] \in \mathbf{H}^s_0 \times H^{s-1}$ such that the solution $[\mathbf{u}, p]$ can be split by*

$$\mathbf{u} = \sum_{i=1}^{2}\mathcal{C}_i\Phi_i + \mathbf{u}_R, \qquad p = \sum_{i=1}^{2}\mathcal{C}_i\phi_i + p_R, \tag{1.4}$$

*and the remainder $[\mathbf{u}_R, p_R]$ and the coefficients $\mathcal{C}_i$ satisfy the regularity estimate:*

$$\mu\|\mathbf{u}_R\|_s + \|p_R\|_{s-1} + \sum_{i=1}^{2}|\mathcal{C}_i| \leq C_b(\|\mathbf{f}\|_{s-2} + \mu\|g\|_{s-1}). \tag{1.5}$$