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Direct computation of stresses in linear elasticity

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ABSTRACT

We present a new finite element method based on the formulation introduced by Philippe G. Ciarlet and Patrick Ciarlet, Jr. in Ciarlet and Ciarlet, Jr. (2005), which approximates strain tensor directly. We also show that the convergence rate of strain tensor is optimal. This work is a non-trivial generalization of its two dimensional analogue in Ciarlet and Ciarlet, Jr. (2009).

1. introduction

This is a continuation of the two-part paper proposed by Ciarlet and Ciarlet, Jr. [1,2]. The main objective of this paper is to introduce and analyze a direct finite element approximation of the minimization problem $j(\underline{\boldsymbol{\varepsilon}}) = \inf_{\underline{\boldsymbol{e}} \in \underline{\boldsymbol{E}}(\Omega)} j(\underline{\boldsymbol{e}})$, which will be introduced in (2.5), to compute stresses inside an elastic body precisely in three dimensional space. The notations and ideas used in this paper to explain our finite element method are due to Ciarlet and Ciarlet, Jr. (see [2]).

Let $\mathbb{S}^3$ denote the space of all 3×3 symmetric matrix. Let Ω be an open, bounded, connected subset of $\mathbb{R}^3$ with Lipschitz boundary. Let $\underline{\boldsymbol{a}} : \underline{\boldsymbol{b}}$ denote the inner product of two matrices $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{b}}$. Now consider a homogeneous, isotropic linearly elastic body with Lame's constants $\lambda > 0$ and $\mu > 0$, with $\overline{\Omega}$ as its reference configuration, and subjected to applied body forces, of density $\boldsymbol{f} \in L^{\frac{6}{5}}(\Omega; \mathbb{R}^3)$ in its interior and of density $\boldsymbol{g} \in L^{\frac{4}{3}}(\Omega; \mathbb{R}^3)$ on its boundary Γ. Given any matrix $\underline{\boldsymbol{e}} = (e_{ij}) \in \mathbb{S}^3$, $A\underline{\boldsymbol{e}} \in \mathbb{S}^3$ is defined by

$$A\underline{\boldsymbol{e}} = \lambda(\mathrm{tr}\underline{\boldsymbol{e}})\mathbf{I}_3 + 2\mu\underline{\boldsymbol{e}}.$$

The classical way to solve pure traction problem of three-dimensional linearized elasticity is to find a displacement vector field $\boldsymbol{u} \in H^1(\Omega; \mathbb{R}^3)$ which satisfies

$$J(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)} J(\boldsymbol{v}),$$

with

$$J(\boldsymbol{v}) = \frac{1}{2}\int_\Omega A \bigtriangledown_s \boldsymbol{v} : \bigtriangledown_s \boldsymbol{v} dx - L(\boldsymbol{v}). \tag{1.1}$$

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For all $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^3)$, where

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v} d\Gamma,$$

and $\bigtriangledown_s \boldsymbol{v}$ denotes the linearized strain tensor field associated with any vector field $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^3)$.

If the applied body forces hold for the compatibility condition $L(\boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \boldsymbol{R}(\Omega)$, where

$$\boldsymbol{R}(\Omega) = \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^3) : \bigtriangledown_s \boldsymbol{v} = 0 \text{ in } \Omega\} = \{\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{x} : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}.$$

Then $\inf_{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3)} J(\boldsymbol{v}) > -\infty$. Besides, thanks to *Korn's inequality*, the solutions of the minimization problem above exist (see [3]) and they are unique up to the addition of any vector field $\boldsymbol{v} \in R(\Omega)$.

Instead of finding vector field $\boldsymbol{u}$ directly, Ciarlet and Ciarlet, Jr. (see [1]) put forward a new method to define a new unknown $\underline{\boldsymbol{e}}$ which is a $d \times d$ symmetric matrix field in d-dimension ($d = 2, 3$). It is proved by Ciarlet and Ciarlet, Jr. in [1] that in both two and three dimensions (note that, Antman [4] already proposed similar idea without proof in 1976 in three-dimensional nonlinear elasticity, while ours is concerned about linear elasticity), if $\underline{\boldsymbol{e}} \in \underline{\boldsymbol{E}}(\Omega)$ ($\underline{\boldsymbol{E}}(\Omega)$ is defined in (2.4)), then any vector field satisfying $\bigtriangledown_s \boldsymbol{v} = \underline{\boldsymbol{e}}$ lies in the set $\{\boldsymbol{v} | \boldsymbol{v} = \dot{\boldsymbol{v}} + \boldsymbol{r} \text{ for some } \boldsymbol{r} \in \boldsymbol{R}(\Omega)\}$ and the mapping $\kappa : \underline{\boldsymbol{E}}(\Omega) \to \dot{\boldsymbol{v}} \in \dot{H}^1(\Omega; \mathbb{R}^3) = H^1(\Omega; \mathbb{R}^3)/\boldsymbol{R}(\Omega)$ such that $\bigtriangledown_s \dot{\boldsymbol{v}} = \underline{\boldsymbol{e}}$ is an isomorphism between $\dot{H}^1(\Omega; \mathbb{R}^3)$ and $\underline{\boldsymbol{E}}(\Omega)$ (see [5–8]). Since the relationship between $\underline{\boldsymbol{e}}$ and $\boldsymbol{v}$ is clear (we choose $\underline{\boldsymbol{e}} = \bigtriangledown_s \boldsymbol{v}$), the minimization problem is converted to:

$$j(\underline{\boldsymbol{\varepsilon}}) = \inf_{\underline{\boldsymbol{e}} \in \underline{\boldsymbol{E}}(\Omega)} j(\underline{\boldsymbol{e}})$$

$$j(\underline{\boldsymbol{e}}) = \frac{1}{2} \int_\Omega A\underline{\boldsymbol{e}} : \underline{\boldsymbol{e}} dx - l(\underline{\boldsymbol{e}}), \tag{1.2}$$

for all $\underline{\boldsymbol{e}} \in \underline{\boldsymbol{E}}(\Omega)$, where

$$l(\underline{\boldsymbol{e}}) = L \circ \kappa,$$

and we can further show that $\underline{\boldsymbol{\varepsilon}} = \bigtriangledown_s \boldsymbol{u}$.

Let Ω be a triangulation of a polyhedral domain in $\mathbb{R}^3$ and $\underline{\boldsymbol{E}}^h$ be a finite element subspace of $\underline{\boldsymbol{E}}(\Omega)$. We consider $\underline{\boldsymbol{e}}^h \in \underline{\boldsymbol{E}}^h$ to be 3×3 symmetric matrix with piecewise constant element. Our goal is to illustrate that how $\underline{\boldsymbol{e}}^h$ can satisfy the condition

$$\mathbf{curl\ curl}\ \underline{\boldsymbol{e}}^h = 0. \tag{1.3}$$

Note that the similar ideas of the mixed finite element methods for linearized elasticity have been discussed in [9–14].

In order to achieve (1.3), the six degrees of freedom that define the elements $\underline{\boldsymbol{e}}^h \in \underline{\boldsymbol{E}}^h$ must be supported by edges (see Lemma 3.1) and they must also satisfy specific compatibility conditions (see Theorem 3.2, Lemma 3.3 and Theorem 3.4). Then the minimization problem can be accomplished. The associated finite elements thus provide examples of edge finite elements in the sense of Nedelec (see [15,16]).

Finally, we consider discrete problem and it naturally comes to find a discrete matrix field $\underline{\boldsymbol{\varepsilon}}^h \in \underline{\boldsymbol{E}}^h$ such that

$$j(\underline{\boldsymbol{\varepsilon}}^h) = \inf_{\underline{\boldsymbol{e}}^h \in \underline{\boldsymbol{E}}^h} j(\underline{\boldsymbol{e}}^h).$$

has a unique solution (see Theorem 4.1), and we establish convergence estimate of the method (see Theorem 4.2).

Note that this approach is the "matrix-analog" of the approximation of the Stokes problem by means of the divergence-free finite elements of Crouzeix and Raviart (although theirs are non-conforming, whereas ours are conforming).

In the following part of the paper, the details of the approach and its validation will be discussed in Section 2. Then the computational space is constructed in Section 3. As shown above, the final section will be the discussion of discrete problem and the convergence estimate of the method.

2. Three dimensional linearized elasticity

The basic notations and conceptions are introduced in Section two of [2]. Let x_i denote the coordinates of a point $\boldsymbol{x} \in R^3$, let $\partial_i := \partial/\partial x_i$ and $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, $i, j \in \{1, 2, 3\}$. Given a smooth enough vector field $\boldsymbol{v}$, we define the 3×3 matrix field $\bigtriangledown \boldsymbol{v} := (\partial_i v_j)$. In this paper, we first consider the domain which is open, bounded and connected subset of $\mathbb{R}^3$ with Lipschitz-continuous boundary. Given any vector field $\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^3)$ as a displacement field, define

$$\bigtriangledown_s \boldsymbol{v} := \frac{1}{2}(\bigtriangledown \boldsymbol{v}^\top + \bigtriangledown \boldsymbol{v}) \in L^2_s(\Omega) := L^2(\Omega; \mathbb{S}^3), \tag{2.1}$$

as its associated symmetrized gradient matrix field. Let

$$\boldsymbol{R}(\Omega) := \{\boldsymbol{r} \in H^1(\Omega; \mathbb{R}^3); \bigtriangledown_s \boldsymbol{r} = 0\} \tag{2.2}$$