Journal of Computational and Applied Mathematics 292 (2016) 417–429

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Construction of efficient general linear methods for stiff Volterra integral equations

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ARTICLE INFO

Article history:
Received 11 November 2014
Received in revised form 25 March 2015

Keywords:
Volterra integral equations
General linear methods
Order conditions
A-stability
V_0-stability
$V_0(\alpha)$-stability

ABSTRACT

General linear methods in the Nordsieck form have been introduced for the numerical solution of Volterra integral equations. In this paper, we introduce general linear methods of order p and stage order $q = p$ for the numerical solution of Volterra integral equations in general form, rather than Nordsieck form. A- and $V_0(\alpha)$-stable methods are constructed and applied on stiff problems to show their efficiency.

1. Introduction

It is the purpose of this paper to investigate the numerical solution of Volterra integral equations (VIEs) of the second kind

$$y(t) = g(t) + \int_{t_0}^{t} K(t, \tau, y(\tau))d\tau, \quad t \in I := [t_0, T], \tag{1}$$

where $g : I \to \mathbb{R}$ is a continuous function and $K : D \times \mathbb{R} \to \mathbb{R}$, with $D := \{(t, \tau) : t_0 \le \tau \le t \le T\}$, is continuous and satisfies the Lipschitz condition with respect to y. Under these assumptions Eq. (1) has a unique continuous solution [1]. On the family of discretization methods for the numerical solution of VIEs, multistage and multivalue methods have been introduced [1–3]. Some of these methods are suitable for solving Eq. (1) in cases where $\partial K/\partial y \ll 0$ or the Lipschitz constant of the kernel with respect to y is large. These integral equations are called stiff [1,4], in analogy with stiff ODEs. To solve stiff VIEs numerically, the applied method must have some reasonably wide region of absolute stability [2,3,5,4]. In this regard, A- and V_0-stable Runge–Kutta methods for VIEs have been introduced in [6–8]. Like general linear methods (GLMs) which have been introduced as a unifying framework for the traditional methods for solving initial value problems [9,10], Izzo et al. [11] investigated the class of GLMs of order p and stage order $q = p$ for the numerical solution of (1) to analyze stability properties of the method conveniently. GLMs combine the essential multivalue and multistage natures of the methods. So, GLMs are multistage–multivalue methods which the classical discretization methods can be considered as special cases of this large family of the methods.

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http://dx.doi.org/10.1016/j.cam.2015.07.028
0377-0427/

In the introduced GLMs in [11], it is assumed that the input and output vectors have the Nordsieck form. In this paper we consider GLMs for the numerical solution of (1) in general form, rather than Nordsieck form. In this way, our aim is to construct A- and $V_0(\alpha)$-stable GLMs of order and stage order $p = q$ to solve stiff VIEs.

Next sections of this paper are organized as follows: In Section 2, we give the representation of GLMs for VIEs in general form. Basic concepts of the methods are discussed in Section 3. In Section 4, the order conditions of the methods with $p = q$ are given. The linear stability is analyzed in Section 5. Section 6 is devoted to construction of examples of A- and $V_0(\alpha)$-stable GLMs of order one and two, and also methods of order three and four with bounded stability regions. Numerical experiments are given in Section 7 to confirm the theoretical results and efficiency of the methods in solving non-stiff and stiff VIEs.

2. Representation of GLMs for VIEs

In this section, we are going to introduce the structure of GLMs for VIEs. Let $t_n = t_0 + nh,\ n = 0, 1, \ldots, N,\ Nh = T - t_0$, be a given uniform grid of I. Then to formulate numerical methods for (1) in $[t_n, t_{n+1}]$, we rewrite this equation in the form

$$y(t) = F^{[n]}(t) + \Phi^{[n+1]}(t), \quad t \in [t_n, t_{n+1}],$$

with the lag term $F^{[n]}(t)$ defined by

$$F^{[n]}(t) = g(t) + \int_{t_0}^{t_n} K(t, \tau, y(\tau))d\tau$$

and the increment term $\Phi^{[n+1]}(t)$ defined by

$$\Phi^{[n+1]}(t) = \int_{t_n}^{t} K(t, \tau, y(\tau))d\tau.$$

A GLM for the numerical solution of VIEs can be characterized by four integers: p the order of method, q the stage order, r the number of external stages and s the number of internal stages.

At the start of step number $n + 1$, denote the input items by $y_i^{[n]}$, $i = 1, 2, \ldots, r$ and the corresponding items output from the step by $y_i^{[n+1]}$. Furthermore, denote the stages computed in the step by $Y_i^{[n+1]}$, $i = 1, 2, \ldots, s$ as an approximation of stage order q to the solution of VIE (1) at $t_{n,i} = t_n + c_i h$, i.e.,

$$Y_i^{[n+1]} = y(t_{n,i}) + O(h^{q+1}).$$

The c_i's represent position of the internal stage within one step. The vector $c = [c_1\, c_2\, \cdots\, c_s]^T$ is called the abscissa vector. The approximations of sufficiently high order to $F^{[n]}(t_{n,i})$ and $\Phi^{[n+1]}(t_{n,i})$ will be denoted by $F_h^{[n]}(t_{n,i})$ and $\Phi_h^{[n+1]}(t_{n,i})$, respectively. For a compact notation, write

$$y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}, \qquad y^{[n+1]} = \begin{bmatrix} y_1^{[n+1]} \\ y_2^{[n+1]} \\ \vdots \\ y_r^{[n+1]} \end{bmatrix}, \qquad Y^{[n+1]} = \begin{bmatrix} Y_1^{[n+1]} \\ Y_2^{[n+1]} \\ \vdots \\ Y_s^{[n+1]} \end{bmatrix},$$

$$F_h^{[n]}(t_{n,c}) = \begin{bmatrix} F_h^{[n]}(t_{n,1}) \\ F_h^{[n]}(t_{n,2}) \\ \vdots \\ F_h^{[n]}(t_{n,s}) \end{bmatrix}, \qquad \Phi_h^{[n+1]}(t_{n,c}) = \begin{bmatrix} \Phi_h^{[n+1]}(t_{n,1}) \\ \Phi_h^{[n+1]}(t_{n,2}) \\ \vdots \\ \Phi_h^{[n+1]}(t_{n,s}) \end{bmatrix}.$$

We now go through the process of carrying out a step in terms of this notation. The formulae for the various steps are

$$\begin{aligned} Y_i^{[n+1]} &= \sum_{j=1}^{s} a_{ij}\left(F_h^{[n]}(t_{n,j}) + \Phi_h^{[n+1]}(t_{n,j})\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, \quad i = 1, 2, \ldots, s, \\ y_i^{[n+1]} &= \sum_{j=1}^{s} b_{ij}\left(F_h^{[n]}(t_{n,j}) + \Phi_h^{[n+1]}(t_{n,j})\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, \quad i = 1, 2, \ldots, r, \end{aligned} \tag{2}$$

or, in the matrix form

$$\begin{aligned} Y^{[n+1]} &= A\left(F_h^{[n]}(t_{n,c}) + \Phi_h^{[n+1]}(t_{n,c})\right) + U y^{[n]}, \\ y^{[n+1]} &= B\left(F_h^{[n]}(t_{n,c}) + \Phi_h^{[n+1]}(t_{n,c})\right) + V y^{[n]}. \end{aligned} \tag{3}$$

Here $A = \left[a_{ij}\right] \in \mathbb{R}^{s\times s}$, $U = \left[u_{ij}\right] \in \mathbb{R}^{s\times r}$, $B = \left[b_{ij}\right] \in \mathbb{R}^{r\times s}$, $V = \left[v_{ij}\right] \in \mathbb{R}^{r\times r}$.