Journal of Computational and Applied Mathematics 292 (2016) 447–468

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Numerical methods for fourth-order elliptic equations with nonlocal boundary conditions

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ARTICLE INFO

Article history:
Received 26 July 2012

MSC:
35J40
35J61
65N06
65N12

Keywords:
Fourth-order elliptic equation
Nonlocal boundary condition
Finite difference system
Method of upper and lower solutions
Monotone iterations
Convergence

ABSTRACT

This paper is concerned with some numerical methods for a fourth-order semilinear elliptic boundary value problem with nonlocal boundary condition. The fourth-order equation is formulated as a coupled system of two second-order equations which are discretized by the finite difference method. Three monotone iterative schemes are presented for the coupled finite difference system using either an upper solution or a lower solution as the initial iteration. These sequences of monotone iterations, called maximal sequence and minimal sequence respectively, yield not only useful computational algorithms but also the existence of a maximal solution and a minimal solution of the finite difference system. Also given is a sufficient condition for the uniqueness of the solution. This uniqueness property and the monotone convergence of the maximal and minimal sequences lead to a reliable and easy to use error estimate for the computed solution. Moreover, the monotone convergence property of the maximal and minimal sequences is used to show the convergence of the maximal and minimal finite difference solutions to the corresponding maximal and minimal solutions of the original continuous system as the mesh size tends to zero. Three numerical examples with different types of nonlinear reaction functions are given. In each example, the true continuous solution is constructed and is used to compare with the computed solution to demonstrate the accuracy and reliability of the monotone iterative schemes.

1. Introduction

There are extensive discussions on fourth-order elliptic equations especially in the area of two-point boundary value problems which arise from the static deflection of a bending beam (cf. [1–3]). However, most of the discussions in the literature are for homogeneous Dirichlet boundary conditions. In a recent article [4] the authors treated a general class of fourth-order elliptic equations with nonlocal boundary conditions by the method of upper and lower solutions and its associated monotone iterations. In this paper, we extend the above method to a corresponding discrete system of the problem and develop various monotone iterative schemes for the computation of numerical solutions, including the

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http://dx.doi.org/10.1016/j.cam.2015.07.018

existence of maximal and minimal solutions, the uniqueness of the solution, and the convergence of the discrete solution to the continuous solution. The basic problem under consideration is given by

$$\begin{aligned} &\Delta^2 u - b_0 \Delta u + c_0 u = f(x, u) \quad (x \in \Omega), \\ &u(x') = \int_\Omega \beta(x', x) u(x) \mathrm{d}x + g^{(1)}(x') \quad (x' \in \partial\Omega), \\ &(\Delta u)(x') = \int_\Omega \beta(x', x)(\Delta u)(x) \mathrm{d}x - g^{(0)}(x') \quad (x' \in \partial\Omega), \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$ $(n = 1, 2, \ldots)$, b_0 and c_0 are constants with $b_0 \ge 0$, and $f(x, u)$, $\beta(x', x)$ and $g^{(l)}(x')$ $(l = 0, 1)$ are continuous functions in their respective domain. The function $\beta(x', x)$ is nonnegative on $\partial\Omega \times \Omega$ and is continuous in x' and is piecewise continuous in x for all $x' \in \partial\Omega$ and $x \in \Omega$. This implies that problem (1.1) is reduced to the standard fourth-order boundary value problem with Dirichlet boundary condition if $\beta(x', x) = 0$ on $\partial\Omega \times \Omega$. By allowing

$$\beta(x', x) = \beta_1 \delta(x - x_1) + \cdots + \beta_p \delta(x - x_p) \quad ((x', x) \in \partial\Omega \times \Omega)$$

for a set of positive constants β_k and points $x_k \in \Omega$, where $k = 1, 2, \ldots, p$ and $\delta(x - x_k)$ is the Dirac function, the boundary condition in (1.1) becomes the multi-point boundary condition

$$u(x') = \sum_{k=1}^{p} \beta_k u(x_k) + g^{(1)}(x'), \qquad (\Delta u)(x') = \sum_{k=1}^{p} \beta_k (\Delta u)(x_k) - g^{(0)}(x'). \tag{1.1$_a$}$$

In our discussion for problem (1.1) we include the above boundary conditions as special cases of the problem.

In addition to the boundary condition in (1.1) we also consider the boundary condition

$$u(x') = \int_\Omega \beta(x', x) u(x) \mathrm{d}x + g^{(1)}(x'), \qquad (\Delta u)(x') = -g^{(0)}(x'). \tag{1.1$_b$}$$

The above boundary condition has been considered in [3,5–9] for the one-dimensional domain $\Omega = (0, 1)$ with $g^{(1)}(x') = g^{(0)}(x') = 0$. The inclusion of possible boundary sources $g^{(1)}(x')$ and $g^{(0)}(x')$ is for the convenience of construction of explicitly known solutions which are used to compare with the computed numerical solutions in the final section.

Fourth-order elliptic boundary value problems have been received considerable attention in recent years, and most of the discussions in the current literature are for the existence, uniqueness and multiplicity of solutions with homogeneous Dirichlet boundary conditions (cf. [10–14]). The works in [15–18] gave also some treatment on the numerical solutions of the corresponding finite difference system, including a second-order elliptic equation with nonlocal boundary condition. On the other hand, extensive attention has been given to ordinary differential equations with either multi-point boundary conditions or integral type of boundary conditions (cf. [7–9,19–25]). The purpose of this paper is to develop some monotone iterative schemes for the computation of numerical solutions of the nonlocal problem (1.1), including the boundary condition (1.1$_b$), using the method of upper and lower solutions. The monotone iterative schemes, called Picard, Gauss–Seidel and Jacobi iterations, yield not only computational algorithms but also existence of maximal and minimal solutions of the corresponding discrete system. These two solutions can be computed by any one of the three iterative schemes using an upper solution or a lower solution as the initial iteration.

The plan of the paper is as follows: In Section 2, we formulate problem (1.1) as a coupled system of two second-order elliptic equations and then discretize it into a finite difference system. A Picard type of monotone iteration is developed for the coupled finite difference system, including the system with boundary condition (1.1$_b$). The Gauss–Seidel and Jacobi monotone iterations as well as a comparison theorem among the three monotone iterations are given in Section 3. In Section 4, we show the convergence of the finite difference solution to the corresponding solution of the continuous problem as the mesh size tends to zero. Section 5 is devoted to the construction of positive upper and lower solutions which are used as initial iterations in the various monotone iterative schemes. Finally in Section 6, we give three examples as applications of the monotone iterations. In each example, an explicitly known continuous solution is constructed and is used to compare with the computed numerical solution to demonstrate the accuracy and reliability of the monotone iterative schemes.

2. Monotone iterations

To develop monotone iterative schemes for numerical solutions of problem (1.1), including the boundary conditions (1.1$_a$) and (1.1$_b$), we discretize the fourth-order equation in (1.1) by the finite difference method. Let h_ν be the spatial increment in the x_ν-direction. Let $i = (i_1, i_2, \ldots, i_n)$ be a multiple index with $i_\nu = 1, 2, \ldots, M_\nu$ for $\nu = 1, 2, \ldots, n$, and let $x_i = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ be an interior point in Ω and $x'_j = (x'_{j_1}, x'_{j_2}, \ldots, x'_{j_n})$ a boundary point on $\partial\Omega$, where M_ν is the total number of interior points in the x_{i_ν}-direction. Denote by Ω_h, $\overline{\Omega}_h$ and $\partial\Omega_h$ the set of mesh points in Ω, $\overline{\Omega}$ and $\partial\Omega$ respectively, and assume that Ω is connected. When no confusion arises we write $i \in \Omega_h$ and $j \in \partial\Omega_h$ instead of $x_i \in \Omega_h$