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# Efficient wavelets-based valuation of synthetic CDO tranches

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ABSTRACT

We present new formulae for the valuation of synthetic collateralized debt obligation (CDO) tranches under a one-factor model. These formulae are based on the wavelet theory and the method used is called $\mathrm{WA}^{[a,b]}$. We approximate the cumulative distribution function (CDF) of the underlying pool by a finite combination of $j$th order B-spline basis, where the B-spline basis of order zero is typically called a Haar basis. We provide an error analysis and we show that for this type of distributions, the rate of convergence in the approximation is similar regardless of the order of the B-spline basis employed. The resulting formula for the Haar basis case is much easier to implement and performs better than the formula for the B-spline basis of order one in terms of computational time. The numerical experiments confirm the impressive speed and accuracy of the $\mathrm{WA}^{[a,b]}$ method equipped with a Haar basis, independently of the inhomogeneity features of the underlying pool. The method appears to be particularly fast for multiple tranche valuation.


## 1. Introduction

The last financial crisis has shown that one of the major source of problems for financial institutions was the credit risk management. The credit derivatives market was the most innovative and fastest growing derivative market during the past ten years. The rapid development was due to new possibilities that were offered by credit derivatives. Credit instruments are flexible financial products that enable the efficient repackaging and transfer of credit risk. Credit derivatives are attractive for yield seeking investors and banks that need to hedge their investments and fulfil the capital requirements. The most popular securities traded on open markets are credit default swaps (CDS), default baskets, and CDOs. A traditional CDO is a credit derivative security whose underlying collateral is a portfolio of risky bonds or bank loans. A synthetic CDO is a credit derivative security whose underlying collateral is a portfolio (or *pool*) made up of CDS. In this paper, we focus our attention on synthetic CDOs and will call them simply CDOs.

To offset the pool owner's risk from these default swaps, a portion of the premiums from them is allocated to a collection of securities called *tranches* of the CDO. There is a priority scheme for the tranches to absorb the pool losses up to fixed maximum amounts for each tranche. Losses are based on the recovery adjusted CDS notional values. The *Equity* tranche is the first to absorb the pool losses. After the Equity tranche is exhausted, losses will affect the *Mezzanine* tranches, and finally the *Senior* tranches. Investors take on exposure to a particular tranche, effectively selling credit protection to the CDO issuer, and in turn, collecting the premium.

Valuation of CDOs is an important problem in credit risk management which requires efficient pricing methods. One issue common to CDO pricing and related risk management is how to evaluate the pool's loss distribution efficiently. From a computational point of view, Monte Carlo simulation (MC) is the last resort because of its inefficiency, despite its flexibility.

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Widely used methods can be divided into two classes. The first class evaluates a pool's loss distribution exactly, based on the assumption that all obligors' losses-given-default sit on a common lattice. Among these methods are the ABS method by [1], the HW method proposed by [2], the LG method developed by [3] and the JKM method by [4]. The ABS and LG methods are directly applicable to inhomogeneous pools. Although the HW and JKM methods are directly applicable to homogeneous pools only, they can be applied indirectly to inhomogeneous pools by noting that in practice an inhomogeneous pool can usually be partitioned into a small number of homogeneous pools. A closed-form solution is derived in [5] in the case of homogeneous instruments. The second class of methods evaluates a pool's loss distribution approximately. An example of this class is the compound Poisson approximation method (CPA) by [6]. In [4] the authors present an improved compound Poisson approximation method based on [7] to enhance the accuracy of the basic CPA [6]. Both the first class of methods and the compound Poisson approximations, strongly rely on the assumption that the loss-given-default must sit on common lattice. To alleviate this deficiency, [4] introduce the normal power (NP) approximation method, widely used in actuarial science, to approximate the pool's loss distribution. However, this approximation works out well for a large pool as a consequence of the central limit theorem, but it may not capture some important properties such as skewness and fat tails. The numerical experiments carried out in [4] on a wide variety of pools, show that JKM method is always faster than HW and much faster than ABS. For most cases JKM is faster than CPA but slower than NP. Another numerical method belonging to the second class of methods is the saddle point (SP) method by [8], which allows the computation of fully inhomogeneous portfolios. It is well known from the literature that SP method fits the tail of the distributions particularly well, however, as pointed out by [9], the method cannot deal properly with highly concentrated portfolios arising from an unequal distribution of the adjusted notional values. This deficiency may affect the Senior tranches in a CDO pricing problem. Yet another approach called EAP method, presented in [10,11], uses a representation of the hockey stick function to directly approximate tranche prices. The authors approximate the payoff function by a sum of exponentials over the positive real line and consequently they do not need to compute the distribution of losses. As pointed out in [11], EAP method is slower than JKM when dealing with very homogeneous pools. The ratio of computation time between multiple tranches evaluation and single tranche evaluation on a single pool, increases with the size of the portfolio for both JKM and EAP method, although it is significantly higher for EAP (see [10] for details). A method based on Laplace transform inversion is presented in [12] within the multifactor model framework.

In this work, we focus on the recovery of the pool's CDF from its characteristic function by means of a wavelets-based method. This method was originally developed within a credit risk environment to recover a CDF on a bounded domain from its Laplace transform by means of a Haar basis (see [13]). The method was extended in [14] to invert Fourier transforms over the entire real line with B-splines up to order one. Later on, it was applied to an option pricing problem in [15] and it was called WA$^{[a,b]}$ method. We aim to use this machinery to efficiently price a CDO, assuming that the correlation structure of default events is described by a one-factor model as in the literature. To this end, we approximate the CDF of the underlying pool first by a finite combination of Haar wavelet functions and second by a finite combination of B-spline wavelets of order one, where the Haar basis can be seen as a B-spline basis of order zero. It is well known that under a one-factor model, the resulting CDF is a staircase like function. For this class of functions, we state a proposition showing that we reach a similar rate of convergence regardless of the order of the B-splines employed. Furthermore, the resulting formula for the Haar basis case is much easier to implement and performs better than the formula for the B-spline basis in terms of computational time. To test this new methodology we carry out the numerical experiments under the particular case of the one-factor Gaussian copula model, although any other one-factor Lévy model can be accommodated (as for example in [16,17]) within this approach. These numerical examples show the robustness and confirm the efficiency of the WA$^{[a,b]}$ method, independently of the inhomogeneity features of the underlying pools. This pricing method is capable to obtain the price of a single tranche of a CDO in about one tenth of a second with a relative error less than one percent. One of the main findings of this work is the ability of this methodology to price multiple tranches of a CDO without adding significant extra computational time, in contrast with JKM and EAP methods.

The rest of the paper is organized as follows. In Section 2 we present the pricing formulae and the default model. In Section 3 we give an overview of some key aspects of the wavelet theory, we briefly explain the WA$^{[a,b]}$ method and we compute the conditional expected cumulative losses. We also provide an error analysis. Section 4 is devoted to the numerical experiments and finally Section 5 concludes.

## 2. The pricing framework

We consider a synthetic CDO tranche of size $S$ with an attachment point $l$, a threshold that determines whether some of the pool losses will be absorbed by this tranche. If the realized losses of the pool are less than $l$, then the tranche will not suffer any loss, otherwise it will absorb losses up to its size $S$. The threshold $S + l$ is called the detachment point of the tranche.

We assume there are $\mathcal{K}$ names in the pool underlying the CDO. For name $\kappa$, its notional value and the recovery rate of the notional value of the reference asset are $N_\kappa$ and $R_\kappa$, respectively. Then the loss-given-default or the recovery adjusted notional value of name $\kappa$ is, $L_\kappa = N_\kappa(1 - R_\kappa)$. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be the set of premium dates, where $T$ denotes the maturity of the CDO tranche. Assume that the interest rates are deterministic. Let $\mathcal{L}_i$ be the pool's cumulative losses up to time $t_i$. Then, the losses absorbed by the specified tranche up to time $t_i$, denoted by $\mathscr{L}_i$ is, $\mathscr{L}_i = g(\mathcal{L}_i; l, S + l) = \min(S, (\mathcal{L}_i - l)^+)$, where $x^+ = \max(x, 0)$.