Journal of Computational and Applied Mathematics 292 (2016) 591–608

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Inverse linear programming with interval coefficients

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ARTICLE INFO

Article history:
Received 20 February 2015
Received in revised form 18 June 2015

MSC:
65G40
90C05

Keywords:
Interval analysis
Interval linear programming
Inverse linear programming
Parametric analysis
Matrix game

ABSTRACT

The paper deals with the inverse linear programming problem over intervals. More precisely, given interval domains for the objective function coefficients and constraint coefficients of a linear program, we ask for which scenario a prescribed optimal value is attained. Using continuity of the optimal value function (under some assumptions), we propose a method based on parametric linear programming techniques. We study special cases when the interval coefficients are situated in the objective function and/or on the right-hand sides of the constraints as well as the generic case when possibly all coefficients are intervals. We also compare our method with the straightforward binary search technique. Finally, we illustrate the theory by an accompanying numerical study, called "Matrix Casino", showing some approaches to designing a matrix game with a prescribed game value.

1. Introduction

Interval linear programming in general. In traditional linear programming (LP) problems, the coefficients of the objective function, of the constraint matrix and of the right-hand sides of the constraints are usually assumed to be known exactly. *Interval linear programming* relaxes this assumption and replaces the data by intervals of possible values. From the practical point of view, the main justification of interval linear programming is that coefficients of LP models are often known only approximately due to elicitation by inexact methods, due to subjective expert evaluations, or due to their inherent vagueness, imprecision or instability. Then it is appropriate to consider intervals of possible values of the coefficients.

When we want to use LP, it is necessary to select representatives of the intervals, for example extreme values or average values. Afterwards, we obtain an optimal solution which is optimal with respect to the chosen representatives; but it is not clear whether the solution is also optimal with respect to the real problem itself. Thus it is often appropriate to take into account all possible choices, instead of the only one determined by the selection of the representatives. Interval linear programming is *the* tool for this issue. Said roughly, interval linear programming is a possibilistic version of linear programming—it takes into account all possible scenarios within given intervals and studies what can happen "in the best and worst case".

A brief review of literature. The first papers dealing with interval LP systematically were [1–3], followed by the state-of-the-art report by Beeck [4]. In the literature, much interest has been devoted to computing the bounds of optimal values, (see [5–11] among others). Determining or enclosing the set of optimal solutions of all the LP problems contained in a

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http://dx.doi.org/10.1016/j.cam.2015.07.034
0377-0427/

family of linear programming problems with interval data was considered in [12,13,4,8,14–18,2,1]. It turned out that the fundamental results of the theory are Oettli–Prager theorem and Gerlach theorem [6]. Their generalization for the case where there is a simple dependence structure between coefficients of an interval system were derived by Hladík [19,7].

The optimal value function. Consider an LP problem in the form

$$\min\{cx \mid Ax = b,\ x \geq 0\} \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The results developed in this paper apply analogously for other LP formulations as well. We denote by

$$f(A, b, c) := \min\{cx \mid Ax = b,\ x \geq 0\}, \tag{2}$$

the optimal objective value of the linear program (1). We also define

$$f(A, b, c) = \begin{cases} +\infty, & \text{if the LP is infeasible,} \\ -\infty, & \text{if the LP is unbounded.} \end{cases}$$

Our goal. In this paper we investigate the optimal value function $f(A, b, c)$ when the entries of A, b and c are subject to independent and simultaneous perturbations in given intervals $\boldsymbol{A} = [A^L, A^U]$, $\boldsymbol{b} = [b^L, b^U]$ and $\boldsymbol{c} = [c^L, c^U]$. Thus, we have a family of LP problems

$$\min\{cx \mid Ax = b,\ x \geq 0\}, \quad A \in \boldsymbol{A},\ b \in \boldsymbol{b},\ c \in \boldsymbol{c}. \tag{3}$$

The family (3) is called *interval linear program*.

From now on, we will use some standard notions from calculus; details can be found in any textbook (for example, the nice book [20] by Rudin).

We will show that under some assumptions, the function $f(A, b, c)$ is continuous. It follows that the optimal value range

$$f(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \{f(A, b, c) : A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}\} \tag{4}$$

is an interval and every value in the interval is attained as the optimal objective value of some problem in the family (3). The main problem of this paper is: we are devoted to finding a concrete problem, called *scenario*, in the family (3) having a prescribed optimal value. To give a precise statement, we solve the problem

$$\begin{aligned} &\textbf{data}: && A^L, A^U \in \mathbb{R}^{m\times n};\ b^L, b^U \in \mathbb{R}^{m\times 1};\ c^L, c^U \in \mathbb{R}^{1\times n};\ \theta \in \mathbb{R} \\ &\textbf{goal}: && \text{find } A_0 \in \mathbb{R}^{m\times n},\ b_0 \in \mathbb{R}^{m\times 1},\ c_0 \in \mathbb{R}^{1\times n} \\ &&& \text{s.t. } \min\{c_0x \mid A_0x = b_0,\ x \geq 0\} = \theta, \\ &&& \quad A^L \leq A_0 \leq A^U,\ \ b^L \leq b_0 \leq b^U,\ \ c^L \leq c_0 \leq c^U, \end{aligned}$$

where the relation $\leq$ between matrices/vectors is understood componentwise.

Applications. This problem enables a decision maker to set up free parameters (here, the coefficients A, b, c of a linear program) to achieve the desired optimal value (which often measures costs or profits). Two examples of applications will be discussed in Sections 5.2 and 5.3. The former example deals with designing a matrix game with a prescribed value and the latter example deals with a problem of determining an optimal fee for playing a game.

We can also mention further situations when it is natural to "tune" the parameters of LPs in given admissible ranges: for example, in the Transportation Problem we can tune the unit transportation prices to achieve a prescribed profit level, or in network flow problems we can tune the capacities of edges to achieve a prescribed value of the maximum flow.

Main results. We present an algorithm based on parametric analysis in LP [21–25]. This provides a new connection between parametric analysis techniques and inverse interval LP problem which is interesting from both computational and theoretical viewpoints. We compare it with a technique based on binary search. Finally, we present an application in designing matrix games. We also refer the reader to the work of Ahmed and Guam [26], which is complementary to ours.

Further remarks. Following [26], we call our approach "Inverse Interval LP" despite ambiguity of the word "inverse" used in optimization. Usually, "inverse optimization" means adjustment of cost coefficients of a given LP problem so that a known feasible solution becomes the optimal one, and the adjustment is minimal in some sense; see [27,28], or [29,30]. The integer programming version of inverse optimization was studied e.g. by Schaefer [31], or Duan and Wang [32]. However, in the interval setting, new interesting questions and problems arise. Hladík [33] proposed a method to compute tolerances for the objective function and constraint coefficients such that the optimal value does not exceed prescribed bounds. Another problem is that one addressed in this paper, that is, to find (A_0, b_0, c_0) in given intervals $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ attaining the prescribed optimal value θ.

2. Preliminaries

Interval matrices. An *interval* $(m \times n)$*-matrix* $\boldsymbol{A} = [A^L, A^U]$ is a family of matrices

$$\{A \in \mathbb{R}^{m\times n} \mid A^L \leq A \leq A^U\},$$