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An iterative method for computing robustness of polynomial stability

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ABSTRACT

We propose a method for computing the distance of a stable polynomial to the set of unstable ones (both in the Hurwitz and in the Schur case). The method is based on the reformulation of the problem as the structured distance to instability of a companion matrix associated to a polynomial. We first introduce the structured ε-pseudospectrum of a companion matrix and write a system of ordinary differential equations which maximize the real part (or the absolute value) of elements of the structured ε-pseudospectrum and then exploit the knowledge of the derivative of the maximizers with respect to ε to devise a quadratically convergent iteration. Furthermore we use a variant of the same ODEs to compute the boundary of structured pseudospectra and compare them to unstructured ones. An extension to constrained perturbations is also considered.

1. Introduction

It is well known that a polynomial is said to be stable either if all its roots lie in the open left half plane (Hurwitz polynomial) or in the unit disk (Schur polynomial). The sensitivity of the roots with respect to perturbations due to data uncertainties and rounding errors has been widely studied during the last decades. In particular, stable polynomials arise very often in control theory where stability has to be intended in Hurwitz (for continuous systems) or Schur (for discrete systems) sense. Due to the presence of parameters or uncertainties, one is often interested in measuring the distance to the nearest unstable system, that is the distance to the nearest polynomial either with a purely imaginary root (Hurwitz case) or with a root with unit modulus (Schur case). More specifically this means that for a polynomial

$$p(z,\delta) = z^n + (c_{n-1} + \delta_{n-1})\, z^{n-1} + \cdots (c_1 + \delta_1)\, z + (c_0 + \delta_0), \quad \text{such that } p(z,0) \text{ is stable}$$

where $\delta = (\delta_{n-1}, \ldots, \delta_0)^{\mathrm{T}}$ denotes the vector of perturbations, we are interested to provide the least upper-bound ρ such that $p(z,\delta)$ remains stable if $\|\delta\| < \rho$.

The topic is widely discussed in the recent literature (see e.g. [1–10], where this short list is far from being exhaustive). For example in the recent monograph [10] several chapters are devoted to the analysis of robust stability, and Chapter 10 deals

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explicitly with the computation of parametric stability margins, that is, of the radii of maximal stability balls in parameter space.

In [1] (based on a theorem of Kharitonov [11]) and [3] the problem is studied respectively for an Hurwitz and a Schur polynomial, with respect to box-shaped perturbations.

In [9] the authors study the robust stability of $p(z, \delta)$ where δ lies in a general convex set of uncertainties on which the coefficients of p depend in an affine way. They prove that the stability robustness of $p(z, \delta)$ can be measured by the maximal nonnegative number ρ with the property that if the Minkowski functional of δ with respect to the convex set is less than ρ, the polynomial $p(z, \delta)$ is always stable. In [8] the authors apply optimization techniques, namely duality results for gauge functions, to derive explicit formulæ for the robust stability radius ρ and in [6] an estimate of the stability radius of a Schur polynomial is determined in terms of Chebyshev polynomials.

In [7] the authors make use of the companion matrix associated to a polynomial and study the effect of perturbations on the coefficients, by pseudospectral tools. In particular they show that for a *balanced* companion matrix the so-called pseudozero set (introduced by Mosier [12]) – that is the set of roots associated to coefficientwise perturbations of the polynomial – and the unstructured pseudospectrum are usually close to each other.

Our approach relies on two grounds. The first is the reformulation of the problem as the one of computing extremal points of a structured pseudospectrum. The second is the appearance of a number of recent iterative algorithms that approximate extremal points of matrix pseudospectra (both structured and unstructured) which are based on the property that extremizers have low-rank (1 or 2 in most problems). The first paper (addressing the computation of extremal points in unstructured pseudospectra) appeared in 2011 [13], followed by a related method in 2012, still for unstructured pseudospectra [14], and extensions to real-structured pseudospectra [15,16] and to spectral value sets [17]. All these methods consider rank-1 or rank-2 continuous or discrete dynamical systems whose stationary points identify local extremizers. Further extensions were considered later for Hamiltonian and symplectic nearness problems (see [18,19]). The power of the proposed techniques is that of being able to treat efficiently large sparse problems and being able to get second order convergence.

Making use of companion matrices, the polynomial stability problem can be reformulated as a structured stability radius problem for a companion matrix (for a general discussion on structured stability radii see [20]). This means that only perturbations that preserve the companion structure are considered in the computation of the stability radius. The concept of stability radius has first been extended in [2] to structured perturbations of a matrix and in [4] the authors give expressions for the stability radius in terms of different norms. The results are derived via state space stability theory of linear systems. A generalization of these results is given in [5].

The connection between stability of a polynomial and the structured ε-pseudospectrum of the associated companion matrix is immediate; when ε is equal to the Hurwitz stability radius, the rightmost point of the structured ε-pseudospectrum lies on the imaginary axis; when ε is equal to the Schur stability radius, the point of largest modulus in the structured ε-pseudospectrum lies on the boundary of the unit disk. A recent method based on a previous result of Toh and Trefethen [7] which characterizes the structured ε-pseudospectrum by an explicit condition is due to [21,22]; it is based on a suitable discretization of a box in the complex plane coupled with a bisection method with respect to ε.

Here we present an iterative method for approximating stability radii based on a two-level iteration (inspired by [16]): an outer one in which the estimate of ε is changed by a Newton procedure and an inner one, where the ε-pseudospectral abscissa or radius (that means points of the structured ε-pseudospectrum with maximal abscissa or modulus) is approximated.

For the latter problem we derive differential equations which allow to compute extremal perturbations, that means perturbations which characterize maximal increase of either the real part of the rightmost root or the root with largest modulus of the polynomial consequent to a perturbation of norm bounded by ε. For the former one, once we have, for a given ε, the ε-pseudospectral abscissa or the ε-pseudospectral radius, we are able to apply Newton's method obtaining quadratic convergence. In general, due to the local optimization behavior of the proposed method, we obtain only an upper bound for the stability radius. Running the iteration starting from different initial points increases the probability of computing the stability radius.

Our method relies on two grounds. The first, as we have mentioned, is the reformulation of the problem as the one of computing extremal points of a structured pseudospectrum. The second is the appearance of a number of recent iterative algorithms that approximate extremal points of matrix pseudospectra (both structured and unstructured) which are based on the property that extremizers have low-rank (1 or 2 in most problems). The first paper (addressing the computation of extremal points in unstructured pseudospectra) appeared in 2011 [13], followed by a related method in 2012, still for unstructured pseudospectra [14], and extensions to real-structured pseudospectra [15,16] and to spectral value sets [17]. All these methods consider rank-1 or rank-2 continuous or discrete dynamical systems whose stationary points identify local extremizers. Further extensions were considered later for Hamiltonian and symplectic nearness problems (see [18,19]). The power of the proposed techniques is that of being able to treat efficiently large sparse problems and being able to get second order convergence.

The paper is organized as follows: in Section 2 we recall some definitions and basic results about companion matrices and structured pseudospectra; in Sections 3 and 4 we derive the differential equations which allow us to find extremizers relative to rightmost points in the structured pseudospectrum of the companion matrix associated to the polynomial, for both complex and real perturbations; in Section 5 we compute the derivative of the pseudospectral abscissa with respect to ε, and describe the method to approximate the (Hurwitz) distance to instability; in Section 6 we extend our method to