Journal of Computational and Applied Mathematics 292 (2016) 785–794

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Path-regularization of linear neutral delay differential equations with several delays

Nicola Guglielmi [a,*], Ernst Hairer [b]

[a] Dipartimento di Ingeneria Scienze Informatiche e Matematica, Università dell'Aquila, via Vetoio (Coppito), I-67010 L'Aquila, Italy

[b] Section de Mathématiques, Université de Genève, 2-4 rue du Lièvre, CH-1211 Genève 4, Switzerland

ARTICLE INFO

Article history:
Received 15 July 2014
Received in revised form 20 November 2014

MSC:
34K40
34K26
65L03

Keywords:
Neutral delay equations
Weak solutions
Discontinuous differential equations
Path-regularization
Stability

ABSTRACT

For differential equations with discontinuous right-hand side and, in particular, for neutral delay equations it may happen that classical solutions do no exist beyond a certain time instant. In this situation, it is common to consider weak solutions of Utkin (Filippov) type. This article extends the concept of weak solutions and proposes a new regularization which eliminates the discontinuities. Codimension-1 and codimension-2 weak solutions are considered. Numerical experiments show the advantages of the new regularization.

1. Introduction

Delay differential equations arise when phenomena with memory are modelled. This article considers systems of linear neutral delay differential equations

$$\begin{aligned} \dot y(t) &= c\big(y(t)\big) + \sum_{j=1}^{m} A_j\big(y(t)\big)\dot y\big(\alpha_j(y(t))\big) \quad \text{for } t>0 \\ y(t) &= \varphi(t) \quad \text{for } t \le 0, \end{aligned} \tag{1}$$

where $y \in \mathbb{R}^n$ and the functions $c(y)$, $A_j(y)$, $\varphi(t)$ and $\alpha_j(y)$ are sufficiently differentiable. We consider time intervals where the solution satisfies $\alpha_j(y(t)) < t$ for all j. The right derivative of the solution of (1) at $t = 0$ is

$$\dot y_0^+ = c\big(\varphi(0)\big) + \sum_{j=1}^{m} A_j\big(\varphi(0)\big)\dot\varphi\big(\alpha_j(\varphi(0))\big), \tag{2}$$

and in general it is different from its left derivative $\dot y_0^- = \dot\varphi(0)$. This produces a jump discontinuity of the derivative $\dot y(t)$ at $t = 0$. Since *neutral* delay equations are considered, the vector field of (1) has a discontinuity when the solution $y(t)$ crosses one of the manifolds given by $\alpha_j(y) = 0$, or evolves in one of them.

* Corresponding author.
E-mail addresses: guglielm@univaq.it (N. Guglielmi), Ernst.Hairer@unige.ch (E. Hairer).

http://dx.doi.org/10.1016/j.cam.2014.12.028

Neutral delay differential equations are closely connected to piecewise smooth dynamical systems (PWS) and in general to ordinary differential equations with discontinuous right-hand side. Excellent monographs on these subjects are those by Filippov [1], Utkin [2] and more recently by Budd, Di Bernardo, Champneys and Kowalczyk [3]. It is well-known that in general ordinary differential equations with discontinuous right-hand-side f do not have a classical solution, and a weak solution concept becomes necessary. A quite popular definition of weak solution is due to Filippov, see [1], who suggested to replace – at a discontinuity point – the right-hand-side by a certain differential inclusion. The resulting method is known as Filippov convexification (a methodology which treats all components of the solution in the same way, as we will see). Similarly when the vector field f is discontinuous due to a discontinuity of one of its arguments, Utkin convexification consists in replacing the discontinuous argument by a convex combination of its left and right limits. Since the source of discontinuity in a neutral delay differential equation of the form $\dot{y}(t) = f(t, y(t), \dot{y}(\alpha(y(t))))$ is a jump of the solution derivative at a past instant, it would seem natural to follow Utkin's approach and replace the discontinuous argument $\dot{y}(\alpha(y(t)))$ by a convex combination of its left and right-hand limits. In the present work, due to the fact the we consider linear problems, Filippov and Utkin convexifications coincide.

However, in presence of a discontinuity hypersurface of codimension bigger than 1, such a convexification becomes ambiguous and a bunch of possible weak solutions arises, so that the contemporary literature has focused its attention on this situation. Recent papers by Dieci, Elia and Lopez have discussed possible ways to retain a unique weak solution. In [4,5] and more recently in [6] the authors have provided a systematic way for defining vector fields on the intersection of several surfaces. Their model passes through the use of a multivalued sign function and is restricted to cases where the sliding manifold is *attractive*.

The possibility of choosing several weak solutions is a main issue of the present paper. An alternative to an a-priori motivated choice for selecting a sliding vector-field, is that of considering regularizations. This idea has also been partially explored in the literature (see e.g. [7–11]) where singularly perturbed smooth systems are proposed to replace the original PWS and analysed.

For neutral delay differential equations the discontinuity is due to a jump at breaking points of the derivative of the solution, which appears on the right hand side of (1) and is determined by the fact that the derivative of the initial function at $t = 0$ is different from the derivative of the solution. Here we consider a so-called path-regularization, which replaces the derivative of the initial datum by a continuous function. Such a path-regularization mainly aims to achieve a few peculiar properties for the regularized solution, as the *closeness* to a classical solution, when it exists, and the absence of high frequency small oscillations, which is a typical drawback (also known as chattering) of introducing singular perturbations in non-smooth systems.

An interesting by-product of our analysis is the appearance of a so-called hidden (or dummy) dynamics (see also [12]), which models the instantaneous behaviour of the discontinuous system at a discontinuity. This is the key point to answer a fundamental question, which involves the (hidden) instantaneous dynamics of a PWS at a discontinuity, which is apparently not described by the system of ODEs.

The present article starts with a summary of previous results (Section 2) and continues with a numerical experiment (Section 3), where the effect of the new path-regularization is illustrated. Section 4 discusses the concept of weak solutions for neutral delay equations. It extends the approach of Filippov [1] and Utkin [2], by allowing more flexibility, which turns out to be essential for the space regularizations introduced in Section 5. Based on singular perturbation techniques [13] the characterization of [14] for the kind of solution, which is approximated by the regularization, is extended to arbitrary paths and to codimension-2 weak solutions. The stability of codimension-2 weak solutions is studied in Section 6 with technical proofs postponed to Section 7. In particular, it is shown that the new path-regularization can avoid highly oscillatory approximations that can be present in the standard space regularization.

2. A summary of previous results

In this section we summarize the main results obtained in the recent articles [14,13], which are concerned with the same kind of problem addressed here. In [14] we have considered neutral state dependent delay differential equations with a *single state-dependent delay* $\alpha(y)$, i.e.

$$\begin{aligned} \dot{y}(t) &= f\big(y(t), \dot{y}\big(\alpha(y(t))\big)\big) \\ y(t) &= \varphi(t) \quad \text{for } t \le 0 \end{aligned} \tag{3}$$

where $y \in \mathbb{R}^n$ and $f(y, z)$, $\varphi(t)$ and $\alpha(y)$ are smooth functions, and we assume $\dot{\varphi}(0) \ne \dot{y}(0^+)$. The possible discontinuities on the right-hand-side, due to jumps in the solution derivative at breaking points, may determine existence termination for the solution. Since such a situation can occur only when the solution meets the manifold $\alpha(y) = 0$ (or $\alpha(y) \in$ {breaking points}), we have to weaken the concept of solution to so-called sliding modes along such a manifold. This is a codimension-1 sliding and is therefore well-described by Filippov and Utkin methodologies. Our goal is not that of applying such methodologies directly but to study suitable regularizations and analyse their limit behaviour (with respect to the regularization parameter ε). In [14] we have considered two regularizations.