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Comparison of a priori and a posteriori meshes for singularly perturbed nonlinear parameterized problems[☆]

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ABSTRACT

This paper deals with the adaptive mesh generation for singularly perturbed nonlinear parameterized problems with a comparative research study on them. We propose an *a posteriori* error estimate for singularly perturbed parameterized problems by moving mesh methods with fixed number of mesh points. The well known *a priori* meshes are compared with the proposed one. The comparison results show that the proposed numerical method is highly effective for the generation of layer adapted *a posteriori* meshes. A numerical experiment of the error behavior on different meshes is carried out to highlight the comparison of the approximated solutions.

1. Introduction

Singular perturbation and its related applications are very common from many prospects like ion transport across biological membranes, pollution dispersion in aqueous media, river flow and financial modeling etc. (see Roos et al. [1]). In general, the solution of singularly perturbed problem exhibits boundary layers. The standard numerical methods on a uniform mesh fail to provide approximate solutions due to the presence of boundary layers. One effective approach is to produce the suitable layer adapted meshes to capture the layer phenomena. These meshes can be divided in two categories.

1. *A priori* meshes: If *a priori* information about the exact solution and its derivatives is available, then this information can be used to construct a suitable layer adapted mesh. This is rarely the case in real life problems.

2. *A posteriori* meshes: First, the error is estimated in terms of an arbitrary mesh and computed solution. Thereafter, in moving mesh context, this estimate will be used to construct a proper layer adapted mesh by a moving mesh algorithm (starting from an initial user chosen mesh). The generation of these meshes is the main goal of this paper.

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In this paper, we consider the following nonlinear singularly perturbed parameterized problem on $\Omega = (0, 1)$ for the *a posteriori* analysis:

$$\begin{cases} Lu \equiv \varepsilon u'(x) + f(x, u(x), \lambda) = 0, \quad x \in \Omega, \\ u(0) = \alpha, \qquad u(1) = \beta, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is the singular perturbation parameter. We assume $f(x, u(x), \lambda) \in C^2(\Omega \times \mathbb{R} \times \mathbb{R})$ with

$$\gamma \le \frac{\partial f}{\partial u} \le \gamma^* \quad \text{and} \quad \delta \le \frac{\partial f}{\partial \lambda} \le \delta^*, \tag{1.2}$$

for some positive constants $\gamma, \gamma^*\ \delta, \delta^*$. Under this condition, the problem (1.1) has a unique solution $(u(x), \lambda)$ (see [2]) on $\overline{\Omega} = [0, 1]$, where $u(x)$ has a boundary layer at $x = 0$.

The existence and uniqueness of the solution for (1.1) is first considered in [3]. Thereafter, it is extended for a general class of parameterized problems in [4]. The numerical analysis of this problem also attracted the research group of Amiraliyev. One can refer the articles [2,5,6] for the efficient numerical methods on *a priori* chosen Shishkin and Bakhvalov meshes. For this problem, the modified upwind scheme on Bakhvalov mesh is suggested by Cen [7]. Numerical analysis based on Runge Kutta method is considered by Xie et al. [8] and Ramos and Vigo-Aguiar [9]. Spectral methods based on rational spectral collocation (for e.g., on Chebyshev points) are also suggested by Wang et al. [10]. In singular perturbation context, the fitted operator methods and Booster method can be seen in [11,12] and [13] respectively. Methods based on the combination of Shishkin and Bakhvalov meshes are considered in [14]. However, the aforementioned methods are all *a priori* and need information about the exact solution which is not available, in general. Here, our main goal is to provide *a posteriori* error estimate for the parameterized singularly perturbed class of problems.

A commonly-used technique in adaptive mesh generation is based on the idea of equidistribution. A mesh $\Omega^N \equiv \{0 = x_0 < x_1 < \cdots < x_N = 1\}$ is said to be equidistributed, if

$$\int_{x_{i-1}}^{x_i} M\big(s, u(s)\big)ds = \frac{1}{N}\int_0^1 M\big(s, u(s)\big)ds, \quad i = 1, \dots, N, \tag{1.3}$$

where $M\big(x, u(x)\big)(> 0)$ is called the monitor function. In this regard, a convergence analysis based on arc length and curvature based monitor functions are provided in [15,16] respectively on the final equidistributed meshes, where mainly the exact solution and the *a priori* information of its derivatives are used. A higher order error estimate by a curvature based error monitor function is also proposed in [17] where de Boor's algorithm [18,19] is used to move the initial mesh points. However, all of these error estimates have extensively used the *a priori* information of the solution and also based on linear singularly perturbed problems. Therefore, the numerical analysis for nonlinear singularly perturbed problems based on *a posteriori* estimate remains a challenged problem till today. This paper is concerned on the *a posteriori* error estimate generation, which will lead to a layer adaptive mesh for (1.1).

The present paper is arranged as follows. First, the stability of the continuous solution is considered in Section 2, which is useful for the proposed *a posteriori* error estimate. Then, we discuss the uniformly convergent results on the *a priori* meshes in Section 3. Thereafter, we establish an *a posteriori* error estimate for the numerical solution. We shall use the distributional derivative concept, wherever required in the analysis. In Section 4, the numerical experiments are carried out to compare the efficiency of the proposed method with the existed methods. Section 5 draws the conclusion of this paper. Throughout this paper, C denotes a generic positive constant independent of ε, x_i and N, which can take different values at different places. We define $\|\cdot\|_\infty$ as $\|\phi\| = \|\phi(x)\|_D = \max_{\eta\in D} |\phi(\eta)|$ for a function ϕ defined on some domain D. If the domain is obvious, we simply write $\|\cdot\|_\infty$ as $\|\cdot\|$.

2. Stability estimate of the continuous solution

This section considers the stability of the continuous solution $(u(x), \lambda)$.

Lemma 2.1. *The continuous solution $(u(x), \lambda)$ of (1.1) satisfies the following inequality*

$$\max(\|u(x)\|, \lambda) \le C, \quad \textit{for } x \in \overline{\Omega},$$

where C is a constant, independent of ε. Moreover, for any $(v(x), \lambda)$ and $(w(x), \mu)$ satisfying $v(0) = w(0)$ and $v(1) = w(1)$ with

$$Lv - Lw = F,$$

where $F(x)$ is a bounded piecewise continuous function, we have

$$\max(\|v - w\|, |\lambda - \mu|) \le C\|Lv - Lw\|,$$

where C is a constant which is independent of ε.