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Order conditions for general linear methods

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ABSTRACT

We describe the derivation of order conditions, without restrictions on stage order, for general linear methods for ordinary differential equations. This derivation is based on the extension of the Albrecht approach proposed in the context of Runge–Kutta and composite and linear cyclic methods. This approach was generalized by Jackiewicz and Tracogna to two-step Runge–Kutta methods, by Jackiewicz and Vermiglio to general linear methods with external stages of different orders, and by Garrappa to some classes of Runge–Kutta methods for Volterra integral equations with weakly singular kernels. This leads to general order conditions for many special cases of general linear methods such as diagonally implicit multistage integration methods, Nordsieck methods, and general linear methods with inherent Runge–Kutta stability. Exact coefficients for several low order methods with some desirable stability properties are presented for illustration.

1. Introduction

Consider the initial value problem for a system of m autonomous ordinary differential equations (ODEs)

$$\begin{cases} y'(x) = f\big(y(x)\big), \quad x \in [x_0, X], \\ y(x_0) = y_0, \end{cases} \tag{1.1}$$

where $y_0 \in \mathbb{R}^m$, and the function $f : \mathbb{R}^m \to \mathbb{R}^m$ is assumed to be sufficiently smooth. Observe that a non-autonomous system of ODEs can always be reduced to an equivalent autonomous system. Let N be a positive integer and define the stepsize $h = (X - x_0)/N$, and the uniform grid $x_n = x_0 + nh$, $n = 0, 1, \ldots, N$. To approximate the solution $y = y(x) \in \mathbb{R}^m$ of the IVP (1.1), we consider the class of multistep-multistage general linear methods (GLMs) introduced by Butcher, and defined more recently in the form in [1] by

$$\begin{cases} Y_i^{[n+1]} = h \sum_{j=1}^{s} a_{ij} f(Y_j^{[n+1]}) + \sum_{j=1}^{r} u_{ij} y_j^{[n]}, \quad i = 1, 2 \ldots, s, \\ y_i^{[n+1]} = h \sum_{j=1}^{s} b_{ij} f(Y_j^{[n+1]}) + \sum_{j=1}^{r} v_{ij} y_j^{[n]}, \quad i = 1, 2 \ldots, r, \end{cases} \tag{1.2}$$

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$n = 0, 1, \ldots, N-1$. At the beginning of each step, it is assumed that approximations $y_i^{[n]} \in \mathbb{R}^m$, $i = 1, 2, \ldots, r$ of (local) order p to some linear combinations of the solution y and its derivatives at the point x_n are available, i.e.,

$$y_i^{[n]} = \sum_{k=0}^{p} q_{ik} h^k y^{(k)}(x_n) + O(h^{p+1}), \quad i = 1, 2, \ldots, r. \tag{1.3}$$

Then, the method computes internal approximations $Y_i^{[n+1]} \in \mathbb{R}^m$ of stage order q to $y(x_n + c_i h)$, i.e.,

$$Y_i^{[n+1]} = y(x_n + c_i h) + O(h^{q+1}), \quad i = 1, 2, \ldots, s, \tag{1.4}$$

where q is the minimum of all stage orders (possibly less than p), and external approximations $y_i^{[n+1]}$ to linear combinations of the solution y and its derivatives at x_{n+1}. The method is defined to be of order p relative to the starting procedure (1.3) if $y_i^{[n+1]}$ has exactly the form of (1.3) when x_n is replaced by x_{n+1}.

The assumption that the starting value is of type (1.3) is stronger than the usual one, and we may be able to find methods with higher order of convergence with respect to more general starting procedures. Different starting procedures are considered in Section 6.3 for G-symplectic methods, and in [1], where elementary differential terms involving partial derivatives of f are present. This class of methods excludes multiderivative methods [2], too. We observe that the restricted consideration to methods which propagate the required solution and its derivatives has advantages for some methods such as a simple strategy for changing step size.

GLMs can be characterized by the abscissa vector $\mathbf{c} = [c_1, \ldots, c_s]^T$, the coefficient matrices

$$\mathbf{A} = [a_{ij}] \in \mathbb{R}^{s\times s}, \qquad \mathbf{U} = [u_{ij}] \in \mathbb{R}^{s\times r}, \qquad \mathbf{B} = [b_{ij}] \in \mathbb{R}^{r\times s}, \qquad \mathbf{V} = [v_{ij}] \in \mathbb{R}^{r\times r},$$

the vectors $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_p \in \mathbb{R}^r$ defined by

$$\mathbf{q}_0 = \begin{bmatrix} q_{1,0} \\ \vdots \\ q_{r,0} \end{bmatrix}, \qquad \mathbf{q}_1 = \begin{bmatrix} q_{1,1} \\ \vdots \\ q_{r,1} \end{bmatrix}, \qquad \ldots, \qquad \mathbf{q}_p = \begin{bmatrix} q_{1,p} \\ \vdots \\ q_{r,p} \end{bmatrix},$$

and the four integers: the order p, the stage order q, the number of external approximations r, and the number of stages or internal approximations s. For more details on stage order and order, see [3, pp. 81–87].

Introducing the notation

$$Y^{[n+1]} = \begin{bmatrix} Y_1^{[n+1]} \\ \vdots \\ Y_s^{[n+1]} \end{bmatrix}, \qquad f\left(Y^{[n+1]}\right) = \begin{bmatrix} f\left(Y_1^{[n+1]}\right) \\ \vdots \\ f\left(Y_s^{[n+1]}\right) \end{bmatrix}, \qquad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix},$$

the GLM (1.2) can be written in a more compact vector form

$$\begin{cases} Y^{[n+1]} = h(\mathbf{A} \otimes \mathbf{I}) f\left(Y^{[n+1]}\right) + (\mathbf{U} \otimes \mathbf{I}) y^{[n]}, \\ y^{[n+1]} = h(\mathbf{B} \otimes \mathbf{I}) f\left(Y^{[n+1]}\right) + (\mathbf{V} \otimes \mathbf{I}) y^{[n]}, \end{cases} \tag{1.5}$$

$n = 0, 1, \ldots, N-1$, where $\mathbf{I}$ is the identity matrix of dimension m and '$\otimes$' stands for the Kronecker product of matrices. (For example, $\mathbf{A} \otimes \mathbf{I}$ is an $sm \times sm$ matrix). Moreover, the relations (1.3) and (1.4) take the form

$$y^{[n]} = z(x_n, h) + O(h^{p+1}), \tag{1.6}$$

and

$$Y^{[n+1]} = y(x_n + \mathbf{c}h) + O(h^{q+1}), \tag{1.7}$$

where the so-called exact value function $z(x, h) \in \mathbb{R}^{rm}$ is defined by

$$z(x, h) = \sum_{k=0}^{p} \mathbf{q}_k h^k y^{(k)}(x) = \mathbf{q_0} y(x) + \mathbf{q}_1 h y'(x) + \cdots + \mathbf{q}_p h^p y^{(p)}(x), \tag{1.8}$$

and

$$y(x + \mathbf{c}h) = \begin{bmatrix} y(x + c_1 h) \\ \vdots \\ y(x + c_s h) \end{bmatrix} \in \mathbb{R}^{rm}.$$