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Control point based exact description of trigonometric/hyperbolic curves, surfaces and volumes

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ABSTRACT

Using the normalized B-bases of vector spaces of trigonometric and hyperbolic polynomials of finite order, we specify control point configurations for the exact description of the zeroth and higher order (mixed partial) derivatives of integral curves and (hybrid) multivariate surfaces determined by coordinate functions that are exclusively given either by traditional trigonometric or hyperbolic polynomials in each of their variables. Based on homogeneous coordinates and central projection, we also propose algorithms for the control point and weight based exact description of the zeroth order (partial) derivative of the rational counterpart of these integral curves and surfaces. The core of the proposed modeling methods relies on basis transformation matrices with entries that can be efficiently obtained by order elevation.

1. Introduction

Normalized B-bases (a comprehensive study of which can be found in [1] and references therein) are normalized totally positive bases that imply optimal shape preserving properties for the representation of curves described as linear combinations of control points and basis functions. Similar to the classical Bernstein polynomials $\left\{B_i^n(u) : u \in [0,1]\right\}_{i=0}^n$ of degree $n \in \mathbb{N}$ – that in fact form the normalized B-basis of the vector space of polynomials of degree at most n on the interval $[0, 1]$, cf. [2] – normalized B-bases provide shape preserving properties like closure for the affine transformations of the control points, convex hull, variation diminishing (which also implies the preservation of convexity of plane control polygons), endpoint interpolation, monotonicity preserving, hodograph and length diminishing, and a recursive corner cutting algorithm (also called B-algorithm) that is the analogue of the de Casteljau algorithm of Bézier curves. Among all normalized totally positive bases of a given vector space of functions a normalized B-basis is the least variation diminishing and the shape of the generated curve more mimics its control polygon. Important curve design algorithms like evaluation, subdivision, degree elevation or knot insertion are in fact corner cutting algorithms that can be treated in a unified way by means of B-algorithms induced by B-bases.

These advantageous properties make B-bases ideal blending function system candidates for curve (and surface) modeling. Using normalized B-basis functions, our objective is to provide control point based exact description (including higher order derivatives) of trigonometric and hyperbolic curves specified with coordinate functions given in traditional parametric form, i.e., in vector spaces

$$\mathbb{T}_{2n}^{\alpha} = \operatorname{span}\mathcal{T}_{2n}^{\alpha} = \operatorname{span}\left\{\cos(ku), \sin(ku) : u \in [0,\alpha]\right\}_{k=0}^{n}, \quad \alpha \in (0,\pi) \tag{1}$$

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or

$$\mathbb{H}_{2n}^{\alpha} = \operatorname{span}\mathcal{H}_{2n}^{\alpha} = \operatorname{span}\left\{\cosh(ku), \sinh(ku) : u \in [0, \alpha]\right\}_{k=0}^{n}, \quad \alpha > 0 \tag{2}$$

where α is a fixed strictly positive shape (or design) parameter. The obtained results will also be extended for the control point based exact description of the rational counterpart of these curves and of multivariate (rational) surfaces that are also specified by coordinate functions given in traditional trigonometric or hyperbolic form along each of their variables.

Concerning the control point based exact description of smooth (rational) trigonometric closed curves and surfaces (i.e., when $\alpha = 2\pi$), Refs. [3,4] already provided control point configurations by using the so-called cyclic basis functions introduced in [5]. Ref. [6] developed a method based on discrete convolution in order to obtain the rational Bézier description of epi- and hypotrochoidal arcs defined on an interval of length strictly less than π. Although, this method can programmatically be extended to calculate the control point based exact description of arcs of (rational) trigonometric curves, the paper does not provide ready to use explicit formulas neither for the evaluation of control points that ensure the exact description of zeroth and higher order derivatives of segments of such curves, nor for the basis transformations between the ordinary and normalized B-basis of the vector space (1).

The rest of the paper is organized as follows. Section 2 briefly recalls some basic properties of rational Bézier curves and points out that curves described as linear combinations of control points and normalized B-basis functions of vector spaces (1) or (2) are in fact special reparametrizations of specific classes of rational Bézier curves. This section also defines control point based integral trigonometric and hyperbolic curves of finite order, briefly reviews some of their properties related to order elevation and at the same time also describes their subdivision algorithm in a slightly more natural way than in [7]. Section 3 provides efficient and parallely implementable recursive formulas for those base changes that transform the normalized B-bases of vector spaces (1) and (2) to their corresponding ordinary (traditional) bases, respectively. Using these transformations, theorems and algorithms of Section 4 provide control point configurations for the exact description of large classes of (rational) trigonometric or hyperbolic curves and multivariate (hybrid) surfaces. All examples included in this section emphasize the applicability and usefulness of the proposed curve and surface modeling tools. In the end, Section 5 closes the paper with our final remarks.

2. Special parametrizations of a class of rational Bézier curves

We will produce control point based exact description of trigonometric and hyperbolic curves, therefore we need proper bases for vector spaces (1) and (2). Whenever it is possible, we will use a unified approach in order to define and formulate the most important (geometric) properties of trigonometric and hyperbolic curves, by using the following notations:

- based on the trigonometric or hyperbolic type of the basis functions consider:
  - the sign parameter $\theta = -1| + 1$;
  - the upper bound $\beta = \pi | + \infty$;
  - the functions $S(u) = \sin(u) \,|\, \sinh(u)$, $S^{-1}(u) = \arcsin(u) \,|\operatorname{arcsinh}(u)$, $C(u) = \cos(u) \,|\, \cosh(u)$, $T(u) = \tan(u) \,|\, \tanh(u)$,

  respectively;
- let $\alpha \in (0, \beta)$ be an arbitrarily fixed design (shape or tension) parameter;
- $\mathcal{F}_{2n}$ corresponds either to the trigonometric or the hyperbolic ordinary basis $\mathcal{T}_{2n}$ or $\mathcal{H}_{2n}$;
- the system

$$\overline{\mathcal{F}}_{2n}^{\alpha} = \left\{F_{2n,i}^{\alpha}(u) = \sigma_{2n,i}^{\alpha} S^{2n-i}\left(\frac{\alpha - u}{2}\right) S^{i}\left(\frac{u}{2}\right) : u \in [0, \alpha]\right\}_{i=0}^{2n} \tag{3}$$

  of order n (degree $2n$) denotes either the linearly reparametrized version of the trigonometric normalized B-basis specified in [8], or the hyperbolic normalized B-basis functions introduced in [9], where the non-negative normalizing coefficients

$$\sigma_{2n,i}^{\alpha} = \frac{1}{S^{2n}\left(\frac{\alpha}{2}\right)} \sum_{r=0}^{\left\lfloor \frac{i}{2} \right\rfloor} \binom{n}{i-r}\binom{i-r}{r}\left(2C\left(\frac{\alpha}{2}\right)\right)^{i-2r}, \quad i = 0, 1, \ldots, n$$

  fulfill the symmetry $\sigma_{2n,i}^{\alpha} = \sigma_{2n,2n-i}^{\alpha}$, $i = 0, 1, \ldots, n$ (in [9], these unique normalizing coefficients are expressed in a different way).

Since the higher order derivatives of ordinary trigonometric and hyperbolic sine and cosine functions behave differently, when it is necessary, we also use the notations

$$\overline{\mathcal{T}}_{2n}^{\alpha} = \left\{T_{2n,i}^{\alpha}(u) = t_{2n,i}^{\alpha} \sin^{2n-i}\left(\frac{\alpha - u}{2}\right) \sin^{i}\left(\frac{u}{2}\right) : u \in [0, \alpha]\right\}_{i=0}^{2n},$$

$$\overline{\mathcal{H}}_{2n}^{\alpha} = \left\{H_{2n,i}^{\alpha}(u) = h_{2n,i}^{\alpha} \sinh^{2n-i}\left(\frac{\alpha - u}{2}\right) \sinh^{i}\left(\frac{u}{2}\right) : u \in [0, \alpha]\right\}_{i=0}^{2n}$$

in order to avoid ambiguity.