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Determining surfaces of revolution from their implicit equations

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Results of number of geometric operations are in many cases surfaces described implicitly. Then it is a challenging task to recognize the type of the obtained surface, find its characteristics and for the rational surfaces compute also their parameterizations. In this contribution we will focus on surfaces of revolution. These objects, widely used in geometric modelling, are generated by rotating a generatrix around a given axis. If the generatrix is an algebraic curve then so is also the resulting surface, described uniquely by a polynomial which can be found by some well-established implicitation technique. However, starting from a polynomial it is not known how to decide if the corresponding algebraic surface is rotational or not. Motivated by this, our goal is to formulate a simple and efficient algorithm whose input is a polynomial with the coefficients from some subfield of $\mathbb{R}$ and the output is the answer whether the shape is a surface of revolution. In the affirmative case we also find the equations of its axis and generatrix. Furthermore, we investigate the problem of rationality and unirationality of surfaces of revolution and show that this question can be efficiently answered discussing the rationality of a certain associated planar curve.

1. Introduction and related work

The choice of a suitable description of a given shape (parametric, or implicit) is fundamental for designing and studying efficient subsequent geometric algorithms in many technical applications. Parameterizations, most often used in Computer-Aided (Geometric) Design, allow us to generate points on curves and surfaces, they are also very suitable for plotting, computing transformations, computing curvatures e.g. for shading and colouring, etc. On the other hand implicit representations are especially suitable for deciding whether a given point is lying on the object, or outside. In addition, it is convenient to intersect two shapes when one is given parametrically and the other implicitly. Finally, in computer graphics, ray tracing is efficiently used for generating an image of implicit algebraic surfaces.

However, we must recall that not every algebraic curve or surface admits a rational parameterization. To be more exact, let $\mathcal{X}$ be a variety over a field $\mathbb{K}$. Then $\mathcal{X}$ is said to be *unirational* if it admits a rational parameterization. Furthermore, if there exists a proper parameterization (i.e., a parameterization with the rational inverse) then $\mathcal{X}$ is called *rational*. By the theorem of Lüroth, a curve has a parameterization if and only if it has a proper parameterization if and only if its genus (see [1] for a definition of this notion) vanishes. Hence, for planar curves the notions of rationality and unirationality are equivalent for

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any field. Algorithmically, the parameterization problem is well-solved, see e.g. [1–3]. In the surface case the theory differs. Over algebraically closed field with characteristic zero, by Castelnuovo's theorem, surface is unirational if and only if it is rational if and only if the arithmetical genus p_a and the second plurigenus P_2 are both zero (see [4] for a definition of these notions). The problem is algorithmically much more difficult than for curves—see e.g. [5] for further details.

The reverse problem (consider a rational parametric description of a curve or a surface, find the corresponding implicit equation) is called the implicitization problem. For any rational parametric curve or surface, we can always convert it into implicit form. Nonetheless, the implicitization always involves relatively complicated process and the resulting implicit form might have large number of coefficients—so, it is not a simple task in general. One can find many generic methods for implicitizing arbitrary rational curves and surfaces such as resultants, Gröbner bases, moving curves and surfaces, and μ-bases—see e.g. [6–9].

In what follows we will deal with implicit surfaces of revolution which are created by rotating a curve around a straight line. Revolution surfaces are well known since ancient times and very common objects in geometric modelling, as they can be found everywhere in nature, in human artefacts, in technical practise and also in mathematics. There has been a thorough previous investigation on finding the implicit equation of a rational surface of revolution. In [10], the authors created an implicit representation for surfaces of revolution by eliminating the square root from $f\left(\sqrt{x^2+y^2}, z\right)$, where $f(x,z)=0$ is the implicit equation of the generatrix curve. Another approach to implicitizing rational surfaces of revolution was presented in [11] where the method of moving planes was efficiently used—the implicit equation of the surface of revolution is then given by the determinant of the matrix whose entries are the $2n$ moving planes that follow the surface, each derived from a distinct 3×3 determinant. A recent technique for implicitizing rational surfaces of revolution was presented in [12]. In this paper, the μ-bases for all the moving planes that follow the surface of revolution were found and subsequently the resultants were used to construct the implicit equation.

In this paper, we will investigate a different interesting problem of computational geometry. We start with an implicit representation and our goal is to decide if the corresponding algebraic surface is rotational or not. Moreover, in case of the positive answer we also want to compute the equations of the axis and the generatrix of the rotational surface. The results of many geometric operations are often described only implicitly. Then it is a challenging task to recognize the type of the obtained surface, find its characteristics and for the rational surfaces compute also their parameterizations. Let us recall e.g. the implicit blend surfaces (often of the canal/pipe/rotational-surface type) offering a good flexibility for designing blends as their shape is not restricted to be constructed as an embedding of a parameter domain. Important contributions for blending by implicitly given surfaces can be found in [13,14]; several methods for constructing implicit blends were thoroughly investigated in [15,16]. Obviously, for choosing a suitable consequent geometric technique is necessary to decide the exact type of the constructed surface. So, the main contribution of this paper is answering the question for the surfaces of revolution which is mentioned in [17] as still unsolved. In addition we will also focus on the question of rationality and unirationality of surfaces of revolution and show that this problem can be efficiently solved transforming it to the question of rationality of a planar curve.

The rest of the paper is organized as follows. In Section 2 we consider an algebraic surface given by equation $f(x,y,z)=0$ for an irreducible polynomial defined over some subfield $\mathbb{K}$ of $\mathbb{R}$, typically $\mathbb{Q}$ or its algebraic extensions. The goal is to decide whether the surface is rotational and eventually to find its axis and profile curve. In this part a symbolic algorithm for recognition of surfaces of revolution is designed and thoroughly discussed. Section 3 deals with the relation between the profile curve (and its quadrat) and the (uni)rationality of the associated surface of revolution. Properties of tubular surfaces are exploited to formulate the results about rationality of surfaces of revolution. Finally we conclude the paper in Section 4. The theory is documented in detail on two computed examples presented in the Appendix.

2. Implicit surfaces of revolution and their recognition

Let be given a straight line $\mathcal{A}$ in Euclidean space $\mathbb{E}^3_{\mathbb{R}}$ and let $\mathcal{G}\subset\mathbb{E}^3_{\mathbb{R}}$ be an algebraic space curve distinct from $\mathcal{A}$. We assume that $\mathcal{G}$ is not a line perpendicular to $\mathcal{A}$. Then the object $\mathcal{X}$ created by rotating $\mathcal{G}$ around $\mathcal{A}$ is an algebraic surface which is called a *surface of revolution* (in what follows, we will write shortly SOR) with the *axis* $\mathcal{A}$ and the *generatrix* $\mathcal{G}$, see Fig. 1 (Left). Assume $\mathcal{X}$ is given by the equation $f(x,y,z)=0$ where $f\in\mathbb{K}[x,y,z]$ for a field $\mathbb{K}$. In addition we consider that $\mathcal{X}$ is absolutely irreducible (i.e., $f\neq f_1\cdot f_2$ for $f_1,f_2\in\mathbb{C}[x,y,z]$).

Of course, there exist a lot of generating curves $\mathcal{G}$ leading to the same surface. Among them we can find one with a prominent role—the *profile curve* $\mathcal{P}$, i.e., the intersection of $\mathcal{X}$ with a plane containing the axis, see Fig. 1 (Right).

In this paper, we want to solve the problem of determining surfaces of revolution from their implicit equations. Our goal is to formulate a simple and efficient symbolic algorithm whose input will be a polynomial with the coefficients from a field and the output will be the decision whether the described algebraic surface is SOR or not. We start with a considerably simpler situation—in particular, we assume that $\mathcal{X}$ is SOR whose axis coincides with the coordinate x-axis. Thus we may obtain its profile curve $\mathcal{P}$ by intersecting $\mathcal{X}$ for instance with the plane $z=0$. Hence, we can consider $\mathcal{P}$ as a curve in xy-plane. Obviously, $\mathcal{P}$ is symmetric with respect to the x-axis. Since $(x,y)\in\mathcal{P}$ if and only if $(x,-y)\in\mathcal{P}$ we deduce that its equation $p(x,y)=f(x,y,0)=0$ can be written in one of the following two forms

$$\sum_i p_i(x)y^{2i}=0,\quad \text{or}\quad \sum_i p_i(x)y^{2i+1}=0. \tag{1}$$