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A power penalty method for second-order cone nonlinear complementarity problems

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ABSTRACT

A power penalty method for solving nonlinear second-order cone complementarity problems (SOCCPs) is proposed. By using this method, the nonlinear SOCCP is converted to asymptotic nonlinear equations. The merit of this method shows that the solution sequence of the asymptotic nonlinear equations converges to the solution of the nonlinear SOCCP at an exponential rate when the penalty parameter tends to positive infinity under mild assumptions. An algorithm is constructed and numerical examples indicate the feasibility of our method.

1. Introduction

In this paper, let $K \subset R^n$ be Cartesian product of second-order cones (SOCs), also called Lorentz cones, i.e.,

$$K = K^{n_1} \times K^{n_2} \times \cdots \times K^{n_m} \tag{1.1}$$

with $m, n_1, \ldots, n_m \geq 1$, $n_1 + \cdots + n_m = n$, and $K^{n_i} \subset R^{n_i}$ being the n_i-dimensional SOC defined by

$$K^{n_i} = \{(x_1, x_2) \in R \times R^{n_i-1} \mid x_1 \geq \|x_2\|\},$$

where $\|\cdot\|$ denotes the Euclidean norm and (x_1, x_2) denotes $(x_1, x_2^T)^T$ for convenience. If $n_i = 1$, K^1 denotes the set of nonnegative reals R_+. Corresponding to the structure of K, we write $x = (x_1, \ldots, x_m)$ with $x_i \in R^{n_i}$, $i = 1, 2, \ldots, m$. The symbols int$K$ and bd$K$ denote the interior of K and the boundary of K respectively. For any $x, y \in R^n$, we denote the partially order relations:

(i) $x \succeq_K y$ (or $y \preceq_K x$) if $x - y \in K$;
(ii) $x \succ_K y$ (or $y \prec_K x$) if $x - y \in \text{int}K$.

Thus, $x \succeq_K 0$ if and only if $x \in K$ and $x \succ_K 0$ if and only if $x \in \text{int}K$.

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Now we consider the second-order cone complementarity problem (SOCCP) (see [1–4]):
Find $x, y \in R^n$ and $\zeta \in R^l$ such that

$$x \in K,\ y \in K,\ \langle x, y\rangle = 0,\ E(x, y, \zeta) = 0, \tag{1.2}$$

where $E : R^n \times R^n \times R^l \to R^n \times R^l$ is a continuously differentiable mapping in general, $\langle \cdot, \cdot\rangle$ denotes the Euclidean inner product, and K is shown in (1.1).

In particular, when $l = 0$ and the mapping E has the form $E(x, y, \zeta) = F(x) - y$ for some nonlinear function $F : R^n \to R^n$, the SOCCP (1.2) becomes the following second-order cone nonlinear complementarity problem (SOCNCP):
Find $x \in R^n$ such that

$$x \in K, F(x) \in K, \langle x, F(x)\rangle = 0, \tag{1.3}$$

where $F = (F_1, \ldots, F_m)$ with $F_i : R^n \to R^{n_i}, i = 1, \ldots, m$. In this paper, we consider F is a continuous function.

The SOCNCP (1.3), in which F does not involve the additional variable ζ, may seem rather restrictive since it is a special case of the SOCCP (1.2). However, for the following second-order cone programming (SOCP):

$$\begin{aligned} &\min\ f(z) \\ &\text{s.t.}\quad g(z) \in \mathcal{K}, \end{aligned} \tag{1.4}$$

where $f : R^s \to R$ and $g : R^s \to R^t$ (including nonlinear functions) are continuously differentiable functions, and $\mathcal{K}$ is some Cartesian product of SOCs, it is worth noting that the Karush–Kuhn–Tucker (KKT) conditions of (1.4) can be written in the form of the SOCNCP (1.3) (see [4]). Particularly, if the SOCP (1.4) is convex problem, then a point z^* is one solution of (1.4) whenever the point satisfies the KKT conditions of the SOCP under appropriate constraint qualification (CQ) (see [5,6]). Therefore, SOCNCP (1.3) is closely related to the convex SOCP. When the mapping F is affine, SOCNCP (1.3) reduces to second-order cone linear complementarity problem (SOCLCP).

During the last two decades, extensive researches have been done for the SOCP and SOCCP for their broad range of applications in engineering design, control, finance, management science, mechanics and economics (see [7–9] and the references therein). Convex SOCP covers linear programs, convex quadratic programs, quadratically constrained convex quadratic programs as well as other related problems. Various numerical methods have been proposed for solving the SOCP and SOCCP, such as the interior-point method (see [7,8,10–13]), the smoothing Newton method (see [1,14–16]), the smoothing-regularization method (see [4]), the semismooth Newton method (see [17,18]), the merit function method (see [19–21]), the matrix splitting method (see [22,23]) and so on. Some approaches, such as [1,4,14,15], require solving the associated Newton equations in which $O(n^3)$ flops are involved, which might be inefficient when the dimension of the problem gets large (see [23]). On the other hand, many complementarity problems require efficient and accurate numerical methods.

The penalty function method is an important method in solving constrained optimization problem. The l_1 exact penalty function and lower order penalty function possess many nice properties and attract much attention (see [24–29]). The smoothing of the exact penalty methods also attracts much attention (see [30–32]). In [33], Wang and Yang proposed a power penalty method (PPM) for solving the linear complementarity problem (LCP), in which the LCP is converted to asymptotic nonlinear equations. The merit of this method shows that the solutions of the asymptotic nonlinear equations converge to the solution of the LCP at an exponential rate when the penalty parameter tends to positive infinity under some mild assumptions. In [34], Huang and Wang extend PPM to solve a class of nonlinear complementarity problem (NCP). The solutions of the asymptotic nonlinear equations also converge to the solution of the NCP at an exponential rate. The power penalty method is actually a kind of lower order penalty method. Due to the well performance of the method in [33,34], the purpose of this paper is to present a power penalty method for solving the SOCNCP under mild assumptions, since such a method for solving the SOCLCP has already obtained in [35]. To the best of our knowledge, there are no advances on the development of PPM for solving the SOCNCP in R^n.

In this work, we convert SOCNCP (1.3) to asymptotic nonlinear equations with penalty parameter σ. Under the assumption that $F(x)$ is ξ-monotone, the SOCNCP (1.3) and nonlinear equations with penalty parameter σ are all unique solvable, where the equivalence between the complementarity problem and some variational inequality plays an important role. We prove that the solutions of asymptotic nonlinear equations converge to the solution of the SOCNCP at an exponential rate $O(\sigma^{-k/\xi})$ when $\sigma \to +\infty$. If the ξ-monotonicity of $F(x)$ is not satisfied, we still show that any limit point of solution sequence solves the SOCNCP (1.3) provided that $F(x)$ is continuous. However, we cannot determine the convergence rate in this situation. The corresponding algorithm is constructed and numerical examples indicate the feasibility of our method.

This paper is organized as follows. Some preliminary results for SOC are presented in the next section. In Section 3, we propose PPM to solve SOCNCP (1.3), the corresponding algorithm is constructed in this section. The convergence analysis for PPM is carried out in Section 4. And in Section 5, numerical results are presented to demonstrate our theoretical findings and we compare the numerical performance of PPM with smoothing Fischer–Burmeister function method. Finally, we give the conclusions.

2. Preliminary results

In this section, we give some preliminary results for a single block SOC, i.e., $K = K^n$, since our analysis can be easily extended to the general case (1.1). For any $x = (x_1, x_2) \in R \times R^{n-1}$, $y = (y_1, y_2) \in R \times R^{n-1}$, we define their *Jordan product*