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Fractional Newton mechanics with conformable fractional derivative

Won Sang Chung

Department of Physics and Research Institute of Natural Science, College of Natural Science, Gyeongsang National University, Jinju 660-701, Republic of Korea

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ABSTRACT

In this paper we use the conformable fractional derivative and integral to discuss the fractional Newtonian mechanics. The fractional version of the calculus of variations is introduced and the fractional Euler–Lagrange equation is constructed. Some mechanical problems such as fractional harmonic oscillator problem, the fractional damped oscillator problem and the forced oscillator problem are discussed in the one-dimensional fractional dynamics.

1. Introduction

Fractional calculus implies the calculus of the differentiation and integration whose order is given by a fractional number. The history of the fractional derivatives goes back to the seventeenth century. There are good textbooks for the fractional calculus [1–7].

During about thirty years or so, fractional calculus has attracted much attention due to its application in various fields of science, engineering, finance and optimal problem. For various applications of fractional calculus in physics, see [8–18] and references therein.

Many applications of fractional calculus amount to replacing the time derivative in an evolution equation with a derivative of a fractional order. One of the problems encountered in the field is what kind of fractional derivative will replace the integer derivative for a given problem. Non-conservative Lagrangian and Hamiltonian mechanics were investigated by Riewe within fractional calculus [19,20]. Besides, Lagrangian and Hamiltonian fractional sequential mechanics, the models with symmetric fractional derivative were studied in [21,22] and the properties of fractional differential forms were introduced [23].

Two types of fractional derivatives, namely Riemann–Liouville and Caputo, are famous. Mathematicians prefer Riemann–Liouville fractional derivative because it is amenable to many mathematical manipulations. However, the Riemann–Liouville fractional derivative of a constant is not zero, and it requires fractional initial conditions which are not generally specified. In contrast, Caputo derivative of a constant is zero, and a fractional differential equation expressed in terms of Caputo fractional derivative requires standard boundary condition. For this reason, physicists and engineers prefer Caputo fractional derivative.

An extension of the simplest fractional variational problem and the fractional variational problem of Lagrange with constraints was obtained by using Caputo fractional derivative [10]. Even more recently, this approach is extended to Lagrangian formalism [24] with linear velocity. The fractional Hamiltonian formulations were presented for discrete and continuous systems whose dynamics were defined in terms of fractional derivative [25,24,26–30].

E-mail address: mimip4444@hanmail.net.

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The Riemann–Liouville derivative and Caputo derivative do not obey the Leibniz rule and chain rule, which sometimes prevent us from applying these derivative to the ordinary physical system with standard Newton derivative. Recently, Khalil, Horani, Yousef and Sababheh introduced the new fractional derivative called conformable fractional derivative and integral [31]. This derivative is well-behaved and obeys the Leibniz rule and chain rule.

In this paper we use the conformable fractional derivative and integral to discuss the fractional version of the Newtonian mechanics. First, from the fractional version of the calculus of variations, we construct the fractional Euler–Lagrange equation. We use this equation to discuss some mechanical problems such as fractional harmonic oscillator problem, the fractional damped oscillator problem and the forced oscillator problem.

2. Fractional calculus of variations

Let us review the conformable fractional calculus [31]. For $0 < \alpha \le 1$, the left conformable fractional derivative (CFD) is defined as

$$D^{\alpha}_{s|x} f(x) = \lim_{\epsilon \to \infty} \frac{f(x + \epsilon (x - s)^{1-\alpha}) - f(x)}{\epsilon} \tag{1}$$

and the right CFD is defined as

$$D^{\alpha}_{x|s'} f(x) = -\lim_{\epsilon \to \infty} \frac{f(x + \epsilon (s' - x)^{1-\alpha}) - f(x)}{\epsilon}. \tag{2}$$

For $0 < \alpha \le 1$, the left conformable fractional integral (CFI) is defined as

$$I^{\alpha}_{s|x} f(x) = \int_s^x (\xi - s)^{\alpha - 1} f(\xi) d\xi \tag{3}$$

and the right CFI is defined as

$$I^{\alpha}_{x|s'} f(x) = \int_x^{s'} (s' - \xi)^{\alpha - 1} f(\xi) d\xi. \tag{4}$$

The CFD and CFI obey the following relations:

$$\begin{aligned} &D^{\alpha}_{s|x} I^{\alpha}_{s|x} f(x) = f(x) \\ &I^{\alpha}_{x|s'} D^{\alpha}_{x|s'} f(x) = f(x) - f(a). \end{aligned} \tag{5}$$

From now on we will restrict ourselves to the case of $s = 0$ and we will denote $D^{\alpha}_{0|x} = D^{\alpha}_x$. Then, the conformable fractional derivative (CFD) is defined as

$$D^{\alpha}_x f(x) = \lim_{\epsilon \to \infty} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon} \tag{6}$$

where $0 < \alpha \le 1$ and D^{α}_x is a right CFD. Using the l'Hospital's rule, we can rewrite the definition of CFD as

$$D^{\alpha}_x f(x) = f'(x) x^{1-\alpha}. \tag{7}$$

The CFD satisfies the following properties:

1. Linearity

$$D^{\alpha}_x (af(x) + bg(x)) = aD^{\alpha}_x f(x) + bD^{\alpha}_x g(x). \tag{8}$$

2. Leibniz rule

$$D^{\alpha}_x (f(x)g(x)) = [D^{\alpha}_x f(x)]g(x) + f(x) D^{\alpha}_x g(x). \tag{9}$$

3. Chain rule

$$D^{\alpha}_x f(g(x)) = [D^{\alpha}_{g(x)} f(g(x))][D^{\alpha}_x g(x)] g(x)^{\alpha - 1}. \tag{10}$$

To derive the fractional version of the Euler–Lagrange equation let us introduce the following fractional functional

$$J^{\alpha}[y] = I^{\alpha}_{0|x_0} f(y(x), D^{\alpha}_x y(x)). \tag{11}$$

We will find the condition that $J^{\alpha}[y]$ has a local minimum. To do so, we consider the new fractional functional depending on the parameter ϵ^{α}

$$J^{\alpha}[\epsilon^{\alpha}] = I^{\alpha}_{0|x_0} f(Y(x, \epsilon^{\alpha}), D^{\alpha}_x Y(x, \epsilon^{\alpha})), \tag{12}$$

where

$$Y(x, \epsilon^{\alpha}) = y(x) + \epsilon^{\alpha} \eta(x) \tag{13}$$

$$D^{\alpha}_x Y(x, \epsilon^{\alpha}) = D_x y(x) + \epsilon^{\alpha} D_x \eta(x) \tag{14}$$