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Extended Gaussian type cubatures for the ball

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ABSTRACT

We construct cubatures that approximate the integral of a function u over the unit ball by the linear combination of surface integrals over the unit sphere of normal derivatives of u and surface integrals of u and $\Delta^2 u$ over m spheres, centered at the origin. We derive explicitly the weights and the nodes of these cubatures, and show that they are exact for all $(2m+2)$-harmonic functions.

1. Introduction

Recently, there has been a substantial effort to extend some of the classical results on quadrature formulas to higher dimensions. New cubature formulas for balls, simplices, spheres and parallelepipeds, based on integrals over low dimensional manifolds, have been suggested, see [1–7], and the references therein. The interest in such cubatures stems from various new technological advances and the need for rigorous mathematical theory to support them. For example, the recent development of the Thermoacoustic Tomography (TT) as one of the promising methods of medical imaging revived the interest in the so-called circular Radon transform, which integrates a function over a set of spheres with a given set of centers. The TT procedure sends a short microwave or radio-frequency electromagnetic pulses through a biological object. At each internal location $\mathbf{x}$, certain energy $u(\mathbf{x})$ is absorbed. The absorbed energy, due to resulting heating, causes thermoelastic expansion, which in turn creates a pressure wave. This wave can be detected by ultrasound transducers placed outside the body. It has been shown that these transducers effectively measure the integrals of u over all spheres centered at the transducers locations. The development of this new technology requires answers to several associated with it mathematical problems, among which are the uniqueness, stability and efficiency in the recovery of u or linear functionals of u from the given data.

In this paper, we deal with the construction and the characterization of several cubatures that approximate the integral of a function u over the unit ball by the linear combination of surface integrals of u over spheres, centered at the origin. More precisely, we construct high dimensional analogues to the result from [8], where Turan's problem [9] of finding a quadrature formula of the form

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{k=1}^{m} (a_k f(x_k) + b_k f''(x_k)), \tag{1.1}$$

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that is exact for all univariate polynomials of degree at least $2m$ has been solved. It was shown in [8] that the quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx \frac{2}{m(m+3)}\sum_{k=1}^{m}\left(\frac{f(x_k)}{P_{m+1}^2(x_k)} + \frac{(1-x_k^2)f''(x_k)}{(m+1)(m+2)P_{m+1}^2(x_k)}\right),$$

where P_{m+1} is the Legendre polynomial and $\{x_i\}_{i=1}^m$ are the zeroes of P'_{m+1}, has algebraic degree of precision $2m+1$. This result has been extended in [10], where the authors present a cubature for approximating the integral of a function u over the unit ball $B := \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| := (\sum_{i=1}^n x_i^2)^{1/2} < 1\}$ in $\mathbb{R}^n$ using, instead of point evaluations, integrals over spheres $S(r_k) := \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| = r_k\}$, namely

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx \sum_{k=1}^{m}\left(A_k \int_{S(r_k)} u(\xi)\,d\sigma(\xi) + B_k \int_{S(r_k)} \Delta u(\xi)\,d\sigma(\xi)\right). \tag{1.2}$$

They showed that formula (1.2) is exact for all polyharmonic functions of order $2m+1$, and explicitly computed the weights $\{A_k\}$, $\{B_k\}$ and the nodes $\{r_k\}$.

Here, we present analogues to cubature (1.2), where we use information along the boundary $S(1)$ of the integration domain B and integrals over spheres $\{S(\tau_j)\}$ of the function and its second order Laplacian Δ^2. To further describe our results, we need some notation. Let us denote by $B(r)$ the Euclidean ball in $\mathbb{R}^n$ with radius r. Recall that a function u, defined on B, is called a *polyharmonic function of order* p (or p-harmonic function), see [11,12], if $u \in C^{2p-1}(\bar{B}) \cap C^{2p}(B)$ and it satisfies the equation

$$\Delta^p u(\mathbf{x}) = 0, \quad \mathbf{x} \in B, \text{ where } \Delta := \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}, \qquad \Delta^p := \Delta(\Delta^{p-1}). \tag{1.3}$$

In particular, when $p = 1$ ($p = 2$), u is called harmonic (biharmonic). The set of all p-harmonic functions on B is denoted by $H^p(B)$. We also denote by $\frac{\partial u}{\partial \nu}$ the normal derivative of u, where ν is the outward unit normal to the sphere $S(1)$.

In this paper, we construct cubature of type I, that is cubature of the form

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx A\int_{S(1)} u(\xi)d\sigma(\xi) + \sum_{j=1}^{m}\left(B_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi) + C_j \int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi)\right), \tag{1.4}$$

and cubature of type II,

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx F\int_{S(1)} \frac{\partial u}{\partial \nu}(\xi)\,d\sigma(\xi) + \sum_{j=1}^{m}\left(G_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi) + H_j \int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi)\right), \tag{1.5}$$

that are exact for all $(2m+2)$-harmonic functions. We call such formulas Lobatto–Turan cubatures. We view them as multidimensional analogues of the classical Lobatto quadratures since they use information along the boundary $S(1)$ of the integration domain B, such as $\int_{S(1)} \frac{\partial u}{\partial \nu}(\xi)\,d\sigma(\xi)$ or $\int_{S(1)} u(\xi)\,d\sigma(\xi)$. They are also multidimensional generalizations of Turan's problem (1.1), because they involve higher order derivatives of the integrand, that is $\left\{\int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi)\right\}_j$.

Since all polynomials in n variables of degree at most $2p-1$ are p-harmonic functions, namely

$$\pi_{2p-1}(\mathbb{R}^n) \subset H^p(B), \tag{1.6}$$

finding multi-dimensional cubature formulas that are exact for $H^p(B)$ for p as large as possible is a natural generalization of the notion of Gaussian quadratures in the one dimensional case. The largest natural number ℓ for which a cubature is exact for all $u \in H^\ell(B)$ is called a *Polyharmonic Degree of Precision* (PDP) of this cubature. Formulas for numerical integration with the best possible PDP are called Gaussian cubatures. In the process of deriving cubatures (1.4) and (1.5), we obtain also a formula of the form

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx E_0\int_{S(1)} u(\xi)d\sigma(\xi) + E_1\int_{S(1)} \frac{\partial}{\partial \nu}u(\xi)\,d\sigma(\xi) + \sum_{j=1}^{m} D_j \int_{S(\tau_j)} u(\xi)\,d\sigma(\xi), \tag{1.7}$$

that has PDP(1.7) $= 2m+2$. We call it the Gauss–Lobatto cubature for the ball because it is a natural generalization of a one dimensional Gauss–Lobatto quadrature, and we show that there are no other cubatures of this form that integrate exactly polyharmonic functions of higher order. We also explicitly construct a cubature of the type

$$\int_B u(\mathbf{x})\,d\mathbf{x} \approx P_0\int_{S(1)} u(\xi)d\sigma(\xi) + P_1\int_{S(1)} \frac{\partial}{\partial \nu}u(\xi)\,d\sigma(\xi) + \sum_{j=1}^{m} Q_j \int_{S(\tau_j)} \Delta^2 u(\xi)\,d\sigma(\xi), \tag{1.8}$$