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Monotone iterates for solving nonlinear integro-parabolic equations of Volterra type

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ABSTRACT

The paper deals with numerical solving of nonlinear integro-parabolic problems based on the method of upper and lower solutions. A monotone iterative method is constructed. Existence and uniqueness of a solution to the nonlinear difference scheme are established. An analysis of convergence rates of the monotone iterative method is given. Construction of initial upper and lower solutions is discussed. Numerical experiments are presented.

1. Introduction

Integro-parabolic differential equations of Volterra type arise in the chemical, physical and engineering sciences (see [1] for details). In this paper we give a numerical treatment for nonlinear integro-parabolic differential equations of Volterra type. The parabolic problem under consideration is given in the form

$$u_t - Lu + f(x,t,u) + \int_0^t g_0(x,t,s,u(x,s))ds = 0, \quad (x,t) \in \omega \times (0,T], \tag{1}$$
$$u(x,t) = h(x,t), \quad (x,t) \in \partial\omega \times (0,T],$$
$$u(x,0) = \psi(x), \quad x \in \overline{\omega},$$

where ω is a connected bounded domain in $\mathbb{R}^\kappa$ ($\kappa = 1, 2, \ldots$) with boundary $\partial\omega$. The linear differential operator L is given by

$$Lu = \sum_{\alpha=1}^{\kappa} \frac{\partial}{\partial x_\alpha}\left(D(x,t)\frac{\partial u}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{\kappa} v_\alpha(x,t)\frac{\partial u}{\partial x_\alpha},$$

where the coefficients of the differential operators are smooth and D is positive in $\overline{\omega} \times [0,T]$. It is also assumed that the functions f, g_0, h and ψ are smooth in their respective domains.

In solving such nonlinear problems by the finite difference or finite element methods, the corresponding discrete problem on each discrete time level is usually formulated as a nonlinear system of algebraic equations. A basic mathematical concern of this problem is whether the nonlinear system possesses a solution. This nonlinear system requires some iterative method for the computation of numerical solutions. This leads to the question of convergence of the sequence of iterations. The aim of this paper is to investigate the above questions concerning the existence and uniqueness of a solution to the nonlinear system, methods of iterations for the computation of the solution.

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Our iterative scheme is based on the method of upper and lower solutions and associated monotone iterates. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem.

Monotone iterative schemes for solving nonlinear parabolic equations were used in [2–8]. In [9], a monotone iterative method for solving nonlinear integro-parabolic equations of Fredholm type is presented. Here, the two important points in investigating the monotone iterative method concerning a stopping criterion on each time level and estimates of convergence rates, in the case of solving linear discrete systems on each time level inexactly, were not given. In this paper, we investigate the monotone iterative method in the case when on each time level nonlinear difference schemes are solved inexactly, and give an analysis of convergence rates of the monotone iterative method.

The structure of the paper as follows. In Section 2, we introduce a nonlinear difference scheme for the numerical solution of (1). A monotone iterative method is presented in Section 3. Existence and uniqueness of the solution to the nonlinear difference scheme are established. An analysis of convergence rates of the monotone iterative method is given. Convergence of the nonlinear difference scheme to the nonlinear integro-parabolic problem (1) is established. Section 4 deals with construction of initial upper and lower solutions. Section 5 presents results of numerical experiments.

2. The nonlinear difference scheme

On the domains $\overline{\omega}$ and $[0, T]$, we introduce meshes $\overline{\omega}^h$ and $\overline{\omega}^\tau$, respectively. For solving (1), consider the nonlinear two-level implicit difference scheme

$$\mathcal{L}U(p,t_k) + f(p,t_k,U) + g(p,t_k,U) - \tau_k^{-1}U(p,t_{k-1}) = 0, \quad (p,t_k) \in \omega^h \times (\overline{\omega}^\tau \setminus \{0\}), \tag{2}$$

with the boundary and initial conditions

$$U(p,t_k) = h(p,t_k), \quad (p,t_k) \in \partial\omega^h \times (\overline{\omega}^\tau \setminus \{0\}), \qquad U(p,0) = \psi(p), \quad p \in \overline{\omega}^h,$$

where $\partial\omega^h$ is the boundary of $\overline{\omega}^h$ and time steps $\tau_k = t_k - t_{k-1},\ k \geq 1, t_0 = 0$.

The difference operator $\mathcal{L}$ is defined by

$$\mathcal{L}U(p,t_k) = \mathcal{L}^h U(p,t_k) + \tau_k^{-1}U(p,t_k),$$
$$\mathcal{L}^h U(p,t_k) = d(p,t_k)U(p,t_k) - \sum_{p' \in \sigma'(p)} a(p',t_k)U(p',t_k),$$

where $\sigma'(p) = \sigma(p) \setminus \{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p \in \omega^h$. We make the following assumptions on the coefficients of the difference operator $\mathcal{L}^h$:

$$d(p,t_k) > 0, \qquad a(p',t_k) \geq 0, \quad p' \in \sigma'(p), \qquad d(p,t_k) - \sum_{p' \in \sigma'(p)} a(p',t_k) \geq 0, \quad (p,t_k) \in \omega^h \times (\overline{\omega}^\tau \setminus \{0\}). \tag{3}$$

The integral g in (1) is approximated by the finite sum g based on the Riemann sum (the rectangular rule)

$$g(p,t_k,U) = \sum_{l=1}^{k} \tau_l g_0(p,t_k,t_l,U(p,t_l)).$$

We also assume that the mesh $\overline{\omega}^h$ is connected. It means that for two interior mesh points $\tilde{p}$ and $\hat{p}$, there exists a finite set of interior mesh points $\{p_1, p_2, \ldots, p_s\}$ such that

$$p_1 \in \sigma'(\tilde{p}),\ p_2 \in \sigma'(p_1), \ldots,\ p_s \in \sigma'(p_{s-1}), \quad \hat{p} \in \sigma'(p_s). \tag{4}$$

On each time level $t_k,\ k \geq 1$, introduce the linear problem

$$(\mathcal{L} + \overline{c})W(p,t_k) = \Psi(p,t_k), \quad p \in \omega^h, \qquad \overline{c}(p,t_k) \geq 0, \qquad W(p,t_k) = h(p,t_k), \quad p \in \partial\omega^h. \tag{5}$$

We now formulate the maximum principle for the difference operator $\mathcal{L} + \overline{c}$ and give an estimate to the solution to (5).

Lemma 1. *Let the coefficients of the difference operator $\mathcal{L}^h$ satisfy (3) and the mesh $\overline{\omega}^h$ be connected.*

(i) *If a mesh function $W(p,t_k)$ satisfies the conditions*

$$(\mathcal{L} + \overline{c})W(p,t_k) \geq 0\ (\leq 0), \quad p \in \omega^h, \qquad W(p,t_k) \geq 0\ (\leq 0), \quad p \in \partial\omega^h,$$

then $W(p,t_k) \geq 0\ (\leq 0)$ in $\overline{\omega}^h$.