Journal of Computational and Applied Mathematics 290 (2015) 239–258

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Orthogonal polynomials of equilibrium measures supported on Cantor sets

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ARTICLE INFO

Article history:
Received 16 October 2014
Received in revised form 16 May 2015

MSC:
42C05
31A15
47B36
81Q10
30C30

Keywords:
Potential theory
Equilibrium measure
Orthogonal polynomials
Conformal mapping
Iterated Function systems
Almost periodic Jacobi matrices

ABSTRACT

The equilibrium measure of a compact set is a fundamental object in logarithmic potential theory. We compute numerically this measure and its orthogonal polynomials, when the compact set is a Cantor set, defined by an Iterated Function System.

We first construct sequences of discrete measures, via the solution of large systems of non-linear equations, that converge weakly to the equilibrium measure. Successively, we compute their Jacobi matrices via standard procedures, enhanced for the scope. Numerical estimates of the convergence rate to the limit Jacobi matrix are provided, which show stability and efficiency of the whole procedure. As a companion result, we also compute Jacobi matrices in two other cases: equilibrium measures on finite sets of intervals, and balanced measures of Iterated Function Systems.

These algorithms can reach large polynomial orders: therefore, we study the asymptotic behavior of the orthogonal polynomials and, by a natural extension of the concept of regular root asymptotics, we derive an efficient scheme for the computation of complex Green's functions and of related conformal mappings.

1. Introduction

1.1. Problem formulation and goals of this paper

Orthogonal polynomials, $\{p_j(\mu; s)\}_{j\in\mathbf{N}}$, of a positive Borel measure μ supported on a compact subset E of the real axis are defined in a straightforward way by the relation $\int p_j(\mu; s)p_m(\mu; s)d\mu(s) = \delta_{jm}$, where δ_{jm} is the Kronecker delta. The well known three-term recurrence relation

$$sp_j(\mu; s) = b_{j+1}p_{j+1}(\mu; s) + a_j p_j(\mu; s) + b_j p_{j-1}(\mu; s), \tag{1}$$

initialized by $b_0 = 0$ and $p_{-1}(\mu; s) = 0$, $p_0(\mu; s) = 1$, can be formally encoded in the Jacobi matrix $J(\mu)$:

$$J(\mu) := \begin{pmatrix} a_0 & b_1 & & \\ b_1 & a_1 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2}$$

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http://dx.doi.org/10.1016/j.cam.2015.05.014

For compact support E the moment problem is determined [1], and the matrix $J(\mu)$ is in one-to-one relation with the measure μ.

While originally introduced for applications (quadratures, optimal control) the rôle of orthogonal polynomials in harmonic analysis, analytical functions and potential theory soon emerged [2,3], and appears clearly in their asymptotic properties for large order, beautifully described in the by-now classical book by Stahl and Totik [4]. These relations are particularly intriguing in the case of measures supported on Cantor sets, the object of this paper. Consider in fact limits such as the *ratio asymptotics* $p_{j+1}(\mu; z)/p_j(\mu; z)$, or the *j-th root asymptotics*, $|p_j(\mu; z)|^{1/j}$, where z is a point in the complex plane and the order j tends to infinity. Under well specified conditions [4], these limits exist and yield the Green's function, $g(E; z)$, of the Dirichlet problem for the complement of the set E [5,6]. In turn, the Green's function is related to an additional measure, that appears in two different forms, ν_μ and ν_E, that may, or may not, exist and coincide.

On the one hand, ν_μ is the *counting measure* of the zeros of the orthogonal polynomials: letting ξ_l^j, for $l = 1, \dots, j$, be the zeros of $p_j(\mu; z)$ and D_x be the unit mass, atomic (Dirac) measure located at the point x, the measure ν_μ is defined by

$$\nu_\mu = \lim_{j\to\infty} \frac{1}{j} \sum_{l=1}^{j} D_{\xi_l^j}, \tag{3}$$

where convergence is meant in the weak * sense.

On the other hand, if $E = \text{supp}(\mu)$ is a compact subset of the complex plane $\mathbf{C}$, ν_E is the *electrostatic equilibrium measure* for a charge distributed on E, with a logarithmic law of repulsion. In fact, let σ be any Borel probability measure, also supported on E. The potential $V(\sigma; z)$, generated by σ at the point z in $\mathbf{C}$, is

$$V(\sigma; z) := -\int_E \log|z - s|\, d\sigma(s). \tag{4}$$

The electrostatic energy $\mathcal{E}(\sigma)$ of the distribution σ is given by the integral of $V(\sigma; z)$:

$$\mathcal{E}(\sigma) := \int_E V(\sigma; u)\, d\sigma(u) = -\int_E \int_E \log|u - s|\, d\sigma(s) d\sigma(u). \tag{5}$$

The *equilibrium measure* ν_E associated with the compact domain E is the unique measure that minimizes the energy $\mathcal{E}(\sigma)$, when this latter is not identically infinite [6,5]. In this case, $\text{Cap}(E) := e^{-\mathcal{E}(\nu_E)}$ defines the capacity of the set E, and the Green's function can be written as

$$g(E; z) = -V(\nu_E; z) - \log(\text{Cap}(E)) = -V(\nu_E; z) + \mathcal{E}(\nu_E). \tag{6}$$

Observe that in the above definition ν_E depends only on the set E. It coincides with ν_μ in the so called *regular* case: measures μ not too thin on any part of their support are regular—see [4] for the precise definition. The measures studied herein will all be regular, so to enable us to use both characterizations, $\nu_\mu = \nu_E$, and the existence of root asymptotics.

Our goal in this paper is to study the equilibrium measure ν_E and its orthogonal polynomials $p_j(\nu_E; z)$. Our main contribution is to devise a reliable computational scheme for the associated Jacobi matrix, $J(\nu_E)$: to the best of our knowledge, this has never been achieved before when E is a Cantor set generated by Iterated Function Systems (IFS) [7–11], to be described in the following. There is a wealth of good reasons to compute the Jacobi matrix of a measure [12,13]; in addition to these, the second main result of this paper is that, when combined with root asymptotics, this leads to an efficient new algorithm for the electrostatic potential $V(\nu_E; z)$, the Green's function $g(E; z)$ and its harmonic conjugate, which also permits to efficiently compute conformal mappings of interest in constructive function theory [14].

1.2. Previous results and motivations for the present work

While many problems in classical orthogonal polynomials have found a complete solution, both from the analytical and the computational viewpoint, much is still to be discovered for non-classical orthogonal polynomials supported on Cantor sets on the real line, despite considerable progress has been made in the last thirty years or so. Among the open problems we think particularly of the phenomenon of intermittency in quantum dynamics [15–20] that can be cast as a problem on the Fourier transforms of orthogonal polynomials of spectral measures.

In fact, the Jacobi matrix $J(\mu)$ can be seen as an operator acting on $l^2(\mathbf{Z}_+)$, the space of square summable sequences. For instance, choosing $b_j = 1/2$ for all j and writing formally $a_j = F(j)$, with F a potential function, yields a *discrete Schrödinger operator* [21–24]. In so doing, μ becomes the *spectral measure* of $J(\mu)$ associated with the first basis vector of $l^2(\mathbf{Z}_+)$, while the equilibrium measure ν_E is known as the *density of states*, a measure which plays a fundamental rôle in many physical properties of the system [25,26].

Quite naturally, the relations between the properties of the two sequences $\{a_j\}_{j\in\mathbf{N}}$, $\{b_j\}_{j\in\mathbf{N}}$, and those of μ have been investigated. For instance, much is known about measures in the Nevai class $N(a, b)$ (*i.e.* those for which $\{a_j\}_{j\in\mathbf{N}}$ and $\{b_j\}_{j\in\mathbf{N}}$ tend to finite limits a and b), as well for Jacobi matrices that are asymptotically periodic [27–29]. Recently, the link between these limits and the classical Szegö asymptotics of orthogonal polynomials has been fully clarified [30,31]. Next in complexity comes the case of discrete Schrödinger operators with *almost periodic* potentials [23,22,32–35,15,16].