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Solving systems of IVPs with discontinuous derivatives—Numerical experiments

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ABSTRACT

We deal in this paper with solving IVPs with singular right-hand side. We consider two recent algorithms of Kacewicz and Przybyłowicz for solving systems of IVPs with right-hand side functions which are globally Lipschitz continuous and piecewise r-smooth with piecewise Hölder rth partial derivatives with Hölder exponent $\rho \in (0, 1]$. The singularity hypersurface is defined by the zeros of an unknown event function. We run several numerical experiments to verify the theoretical results of Kacewicz and Przybyłowicz. Our tests confirm that the bounds on the error $O(n^{-(r+\rho)})$ can be achieved with $O(n)$ function evaluations, where n is a number of discretization points.

1. Introduction

The solution of initial value problems is well studied for regular problems (see e.g. [1–5]). However, in many practical problems that appear the right-hand side functions are only piecewise regular (see e.g.[6–8]). The number of papers dealing with the right-hand side functions whose smoothness drop on some hypersurface is limited. Beginning with the work of Chartres and Stepleman [9] and Mannshardt [10], then Gear and Østerby [11] to the papers of Dieci and Lopez [6,7].

Recently appeared also the papers of Kacewicz and Przybyłowicz dealing with the piecewise regular IVPs [12–15]. The first paper considers the autonomous scalar equations only. The second one deals with scalar non-autonomous equations with separated variables. The most recent papers of Kacewicz and Przybyłowicz [14,15] consider systems of initial value problems with right-hand side functions where regularity drops on certain unknown switching hypersurfaces. It is assumed that a hypersurface is defined by the zeros of an unknown event function. The access to the event function is not allowed even through its values. The rigorous analysis of the bounds on the error and the cost is presented. However, in that papers the numerical tests are not presented.

We present in this paper a number of numerical experiments to verify the theoretical results of [14,15]. We propose examples which satisfy the regularity assumptions imposed in [14,15] on right-hand side functions and also test problems that do not satisfy some of these assumptions. Our tests confirm the theoretical results of Kacewicz and Przybyłowicz. The tests also suggest that these algorithms may work properly even when the right-hand side function is not Lipschitz continuous.

The paper is organized as follows. In Section 2 the problem is formulated and necessary definitions are presented. In Section 3 we recall after [14,15] the algorithms. Section 4 is the main part of the paper and it contains results of numerical experiments. Final remarks and comments are given in Section 5.

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2. Problem formulation

We consider in this paper the initial value problem from [14,15] of the following form

$$z'(t) = f(t, z(t)), \quad t \in [a, b], \qquad z(a) = \eta, \tag{1}$$

where $a < b, f : [a, b] \times \mathbf{R}^d \to \mathbf{R}^d$, $\eta \in \mathbf{R}^d$. Function f is assumed to be piecewise regular, where regularity breaks down on a hypersurface defined by the zeros of the *event function* $h : [a, b] \times \mathbf{R} \to \mathbf{R}$. The event function h is assumed to belong to the class H defined by

$$H = \{h : [a, b] \times \mathbf{R}^d \to \mathbf{R} \mid h \in C^1([a, b] \times \mathbf{R}^d),\ |h(t, y_1) - h(t, y_2)| \le M_0\|y_1 - y_2\|,$$
$$|\partial^1 h(t, y_1) - \partial^1 h(t, y_2)| \le M_1\|y_1 - y_2\|,\ t \in [a, b],\ y_1, y_2 \in \mathbf{R}^d\}.$$

Here we denote by $\partial^1 h$ all partial derivatives of h of the first order, M_0, M_1 are positive constants, $\|\cdot\|$ is the maximum norm of vectors.

We first define a class $F_{reg}^{r,\rho}$ of regular functions. Let $r \ge 0$, $\rho \in (0, 1]$ and $D_0, D_1, \dots, D_r, L, L_r$ be positive numbers. We define

$$F_{reg}^{r,\rho} = \{f : [a, b] \times \mathbf{R}^d \to \mathbf{R}^d \mid f \in C^r([a, b] \times\ \mathbf{R}^d), |\partial^i f_j(t_1, y_1)| \le D_i,\ i = 0, 1, \dots, r,$$
$$|\partial^r f_j(t_1, y_1) - \partial^r f_j(t_2, y_2)| \le L_r(\|y_1 - y_2\|^\rho + |t_1 - t_2|^\rho), |f_j(t_1, y_1) - f_j(t_2, y_2)| \le L(\|y_1 - y_2\| + |t_1 - t_2|),$$
$$t_1, t_2 \in [a, b],\ y_1, y_2 \in \mathbf{R}^d,\ j = 1, 2, \dots, d\}.$$

We denote by $\partial^i f_j$ all partial derivatives of order i of f_j, where $f = [f_1, f_2, \dots, f_d]^T$. It is also assumed that $r + \rho \ge 1$.

We are ready to define a class of piecewise regular right-hand side functions f. Let L and G be positive constants. It is assumed that a function f satisfies the Lipschitz condition on the whole domain,

$$\|f(t_1, y_1) - f(t_2, y_2)\| \le L(\|y_1 - y_2\| + |t_1 - t_2|) \quad \text{for all } t_1, t_2 \in [a, b],\ y_1, y_2 \in \mathbf{R}^d,$$

there exist an event function $h \in H$ and $f_+, f_- \in F_{reg}^{r,\rho}$ such that for all $t \in [a, b]$, $y \in \mathbf{R}^d$

$$f(t, y) = \begin{cases} f_+(t, y), & \text{if } h(t, y) \ge 0, \\ f_-(t, y), & \text{if } h(t, y) < 0, \end{cases}$$

and functions f and h satisfy the *transversality condition* of the form

$$\frac{\partial h}{\partial t}(t, y) + \frac{\partial h}{\partial y}(t, y) f(t, y) \ge G \quad \text{for all } t \in [a, b],\ y \in \mathbf{R}^d. \tag{2}$$

We denote the class of such right-hand side functions f by $F^{r,\rho}$. The numbers $r, \rho, D_0, D_1, \dots, D_r, L, L_r, M_0, M_1, G$ together with a, b, and d are the parameters of the class $F^{r,\rho}$. Except for a, b, d, r and ρ the parameters are not known and the algorithms presented later on will not use them as input parameters. The class $F^{r,\rho}$ consists of functions that are regular everywhere in $[a, b] \times \mathbf{R}^d$ except for the smooth hypersurface $h(t, y) = 0$. It is also assumed that the event function h is unknown and its values cannot be computed. We have only access to the values of the right-hand side function f and its partial derivatives. The possibility of evaluating f does not mean, however, that we are able to obtain values of f_- or f_+.

The transversality condition (2) assures that for any solution $z = z(t)$ of (1), the mapping $[a, b] \ni t \mapsto h(t, z(t))$ is strictly increasing with the first derivative bounded away from zero. This guarantees that $(t, z(t))$ hits the hypersurface $h(t, y) = 0$ in at most one point. We call this point the *transition point* for z.

Let $l : [a, b] \to \mathbf{R}^d$ be a bounded approximation of the solution z of (1). The error for $f \in F^{r,\rho}$ of the approximation l is defined by

$$\sup_{t\in[a,b]} \|z(t) - l(t)\|.$$

The worst-case error of l is defined by

$$\sup_{f\in F^{r,\rho}} \sup_{t\in[a,b]} \|z(t) - l(t)\|.$$

In [14,15] two adaptive algorithms solving (1) are presented. The first algorithm uses the values of a right-hand side function and also the values of its partial derivatives while the latter uses the right-hand side function values only. For both algorithms it is proved that for the discretization parameter $n \in \mathbf{N}$, where $n + 1$ is the number of (equidistant) mesh points in $[a, b]$, they use at most $O(n)$ function evaluations and their error is of order $O(n^{-(r+\rho)})$ (for the first algorithm it is assumed that $\rho = 1$, but this result may by straightforwardly generalized for any $\rho \in (0, 1]$). We will present these algorithms in the following section. Our aim is to confirm these results in a series of numerical tests.