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Stability of differentially heated flow from a rotating sphere

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HIGHLIGHTS

- Differentially heated flow of a thin fluid layer from a rotating sphere has been investigated.
- A numerical solution procedure for solving the steady and unsteady equations has been proposed.
- An approximate analytical solution has been derived.
- A linear stability analysis has estimated a theoretical value for the onset of instability.
- Good agreement was found between numerical, analytical and theoretical results.

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ABSTRACT

We present results on the flow of a thin fluid layer over a rotating sphere having a surface temperature that varies in the latitudinal direction. The fluid is taken to be viscous, incompressible and Newtonian while the flow is assumed to possess both azimuthal and equatorial symmetry. The governing Navier–Stokes and energy equations are formulated in terms of a stream function and vorticity and are solved subject to no-slip boundary conditions. An approximate analytical solution for the steady-state flow has been derived and is compared with numerical solutions to the steady and limiting unsteady equations. For small Rayleigh numbers these solutions are found to be in close agreement. However, as the Rayleigh number is increased noticeable differences occur. A numerical solution procedure is presented and a linear stability analysis has been conducted to predict the onset of instability. Good agreement between the theoretical predictions and the observed numerical simulations was found.

1. Introduction

The flow and heat transfer of a thin fluid layer from a differentially heated rotating sphere are of interest in geophysical and meteorological applications such as weather prediction and climate modelling [1,2]. Differential heating and rotation combined with stratification of the fluid layer make this a challenging problem. One goal of this research is to present a simple mathematical model to describe such flows; another is to construct numerical and analytical solution procedures to solve this problem. A compact mathematical model together with an efficient solver can be used as a platform to explore and better understand model sensitivity to physical processes and discretization [3]. It can also be used to examine, evaluate, and when necessary, revise parameterizations that are currently used in weather forecasting and climate modelling to describe key unresolved sub-grid processes [4].

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Some previous related studies include Marcus and Tuckerman [5,6] who carried out numerical simulations of the flow between two concentric differentially rotating spheres in the absence of heating. In this case the flow is analogous to Taylor–Couette flow [7] whereby Taylor vortices resembling Hadley cells can form. Hart, Glatzmaier and Toomre [8], on the other hand, presented three-dimensional numerical simulations of thermal convection in a rotating hemispherical shell and presented a thorough summary of earlier investigations. The research carried out by Lesueur et al. [9] numerically investigated the flow in a thick spherical shell using parameters that are consistent with those of the outer planets. The fluid was subjected to both internal and external heat sources with the external source representing solar heating. The flow pattern consisted of two Hadley cells on each side of the equator extending from the equator to the poles occupying the entire fluid layer.

Regarding the stability of the flow, one of the earliest works can be attributed to Chandrasekhar [10] who formulated the problem using a spherical shell geometry. Walton [11] extended the classical Rayleigh–Bénard problem [12–14] to include a slowly varying temperature along the bottom plate. His analysis showed that a slowly varying temperature has a stabilizing effect on the flow and hence delays the onset of instability. In this research we have applied his analysis to our problem. Later, Soward and Jones [15] analytically investigated the stability of the isothermal flow in the narrow gap between two concentric differentially rotating spheres. The more recent work of Lewis and Langford [16] blends bifurcation theory and computations to examine the stability of differentially heated flow from a rotating sphere. Their calculations showed that as the equator-to-pole temperature difference increases from zero, large Hadley cells extending from the equator to poles form immediately. As the temperature difference increases two or three convection cells appear in each hemisphere. A related problem involves the stability of a boundary layer on a rotating sphere. Barrow, Garrett and Peake [17] and Garrett and Peake [18], among others, have solved the stability problem for the case of a sphere rotating in a fluid which is otherwise at rest.

Although this study tackles a problem that has been addressed by several researchers, the adopted approach is significantly different. Key differences lie in the mathematical formulation of the problem as well as the numerical and analytical solution procedures. The paper is structured as follows. In Section 2 we present the governing equations and the corresponding initial and boundary conditions. Following that, in Section 3, we conduct a linear stability analysis to estimate the onset of instability. This involves deriving an approximate analytical solution to the steady-state equations which is then fed into the stability analysis. The numerical solution procedure for solving both the steady and unsteady equations is outlined in Section 4. The numerical and analytical results are presented and discussed in Section 5. The investigation is summarized in the concluding section. Lastly, an Appendix A is included to provide more details on the mathematical formulation of the problem.

2. Governing equations

The atmosphere can be thought of as a thin fluid layer covering the surface of a rotating sphere. Based on this a simple mathematical model describing the unsteady laminar convective flow of a viscous incompressible dry Boussinesq fluid from a solid impermeable rotating differentially heated sphere can be formulated. The flow domain and configuration are illustrated in Fig. 1. In the rotating reference frame the fluid is taken to be initially at rest and is then set into motion by buoyancy as a result of a prescribed poleward decrease in surface temperature as well as a radial decrease in temperature. The poleward decrease in surface temperature mimics solar heating and is taken to be sinusoidal since this is consistent with the amount of solar radiation penetrating into the surface as it tilts away from the equator. In addition, the flow is assumed to possess both azimuthal and equatorial symmetry.

Owing to the assumed symmetry, the unsteady Navier–Stokes equations can be expressed in terms of a stream function, ψ, scaled vorticity, ω, and scaled zonal velocity, W. Appendix A presents the primitive formulation of the problem in terms of the velocity components v_r, v_θ, v_ϕ and pressure P, and also explains how the equations for ψ, ω and W can be obtained. In spherical coordinates (r, θ, ϕ) and cast in dimensionless form the governing Navier–Stokes and energy equations can be expressed as

$$\omega = -\delta D^2\psi, \tag{1}$$

$$\frac{\partial \omega}{\partial t} + \frac{\delta}{r^2 \sin\theta}\frac{\partial(\psi,\omega)}{\partial(\theta,r)} + \delta\, Pr\, Ra \sin\theta \frac{\partial T}{\partial \theta} + \frac{2\delta\omega}{r^2\sin^2\theta}\left(\cos\theta\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right)$$
$$- \left(\frac{2\delta^2 W}{r^2\sin^2\theta} + \frac{2\delta^2}{Ro}\right)\left(\cos\theta\frac{\partial W}{\partial r} - \frac{\sin\theta}{r}\frac{\partial W}{\partial\theta}\right) = \delta^2\, Pr\, D^2\omega, \tag{2}$$

$$\delta^2\, Pr\, D^2 W - \frac{\partial W}{\partial t} = \frac{\delta}{r^2\sin\theta}\frac{\partial(\psi, W)}{\partial(\theta, r)} - \frac{2\delta}{Ro}\left(\cos\theta\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right), \tag{3}$$

$$\frac{\partial T}{\partial t} + \frac{\delta}{r^2\sin\theta}\frac{\partial(\psi, T)}{\partial(\theta, r)} = \delta^2\nabla^2 T. \tag{4}$$

As noted in the Appendix A, the key underlying assumptions and approximations made in deriving Eqs. (1)–(4) include: the Boussinesq approximation, ignoring the variation in the gravitational acceleration with radial distance from the surface, and assuming that the rate of rotation is sufficiently small that the centrifugal acceleration, and its impact on pressure, can be