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The majorant method in the theory of Newton–Kantorovich approximations and generalized Lipschitz conditions

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ABSTRACT

We provide a semilocal as well as a local convergence analysis for Newton's and modified Newton's methods in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. We use more precise majorizing sequences than in earlier studies such as Appell et al. (1997), Appell et al. (1991), Argyros (2004), Argyros and Hilout (2009), Kantorovich and Akilov (1982), Ortega and Rheinboldt (1970) and generalized Lipschitz continuity conditions. Our sufficient convergence conditions are weaker than before and our convergence analysis is tighter. Special cases and numerical examples are also given in this study.

1. Introduction

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and let $\mathcal{D} \subseteq \mathcal{X}$ be closed and convex. In the present paper we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where, $F : \mathcal{D} \longrightarrow \mathcal{Y}$ is Fréchet-differentiable.

Many problems from Computational Sciences can be brought in the form of Eq. (1.1) using mathematical modeling [1–3]. The solution of these equations can rarely be found in closed form. Therefore the solution methods for these equations are usually iterative. Note that in Computational Sciences, the practice of numerical analysis for finding such solutions is essentially connected to Newton-type methods [1,2,4,5].

Newton's method is defined by

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad \text{for each } n = 0, 1, 2, \ldots, \tag{1.2}$$

where $x_0 \in \mathcal{D}$ is an initial point. Linear operator $F'(x)$ denotes the Fréchet-derivative of F at $x \in \mathcal{D}$. Newton's method is undoubtedly the most popular iterative process for generating a sequence $\{x_n\}$ approximating $x^\star$. This method requires

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the inversion of linear operator $F'(x)$ as well as the computation of the function F at $x = x_n$ ($n \in \mathbb{N}$) at each step. If x_0 is close enough to $x^\star$, Newton's method converges quadratically [1,4]. The inversion of $F'(x)$ ($x \in \mathcal{D}$) at each step may be too expensive or unavailable. That is why the modified Newton's method

$$y_{n+1} = y_n - F'(y_0)^{-1} F(y_n) \quad \text{for each } n = 0, 1, 2, \ldots \ (y_0 = x_0 \in \mathcal{D}) \tag{1.3}$$

can be used in this case instead of Newton's method. However, the convergence is only linear [6,7,1,4]. The background on the convergence of Newton's and modified Newton's methods can be found in [8–10,6,11,12,7,1,13,14,2,15–18,3,19–27,4,5,28–48].

The study about convergence matter of Newton's or modified Newton's method is usually centered on two types: semilocal and local convergence analyses. The semilocal convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of Newton's method; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There is a plethora of studies on the weakness and/or extension of the hypothesis made on the underlying operators; see for examples [1–5] and the references therein. Concerning the semilocal convergence of Newton's method, one of the most important results is the celebrated Kantorovich theorem for solving nonlinear equations. This theorem provides a simple and transparent convergence criterion for operators with bounded second derivatives F'' or the Lipschitz continuous first derivatives. The second type analysis for numerical methods is the local convergence. Traub and Woźniakowsi [44], Rheinboldt [41,42], Rall [40], Argyros [1] and other authors gave estimates of the radii of local convergence balls when the Fréchet-derivatives are Lipschitz continuous around a solution.

Concerning the semilocal convergence of both methods, the Lipschitz-type condition

$$\|F'(x_0)^{-1} (F'(x) - F'(y))\| \leq \omega(\|x - y\|) \quad \text{for each } x, y \in \mathcal{D} \tag{1.4}$$

has been used [9,10,1,20,22–25,32–39,46–48], where ω is a strictly increasing and continuous function with

$$\omega(0) = 0. \tag{1.5}$$

If $\omega(t) = L t$ for $t \geq 0$, we obtain the Lipschitz case, whereas if $\omega(t) = L t^\mu$ for $t \geq 0$ and fixed $\mu \in [0, 1)$, we obtain the Hölder case. Sufficient convergence criteria in the above references as well as error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$ for each $n = 0, 1, 2, \ldots$ have been established using the majorizing sequence $\{u_n\}$ given by

$$\begin{aligned} &u_0 = 0, \qquad u_1 = \eta > 0, \\ &u_{n+2} = u_{n+1} + \frac{\int_0^1 \omega(\theta\,(u_{n+1} - u_n))\, d\theta\, (u_{n+1} - u_n)}{1 - \omega(u_{n+1})} \quad \text{for each } n = 0, 1, 2, \ldots. \end{aligned} \tag{1.6}$$

Using (1.6) (see [1,9,10]) we have that

$$u_{n+2} \leq u_{n+1} + \frac{\chi(u_{n+1})}{1 - \omega(u_{n+1})} \quad \text{for each } n = 0, 1, 2, \ldots$$

where,

$$\chi(t) = \eta - t + \int_0^1 \overline{\omega}(r)\, dr \quad \text{and} \quad \overline{\omega}(t) = \sup_{t_1 + t_2 = t} (\omega(t_1) + \omega(t_2)).$$

Under the same or weaker convergence criteria, we provided a convergence analysis [6,1,17,3] with the following advantages over the earlier stated works: tighter error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$ for each $n = 0, 1, 2, \ldots$ and an at least as precise information on the location of the solution $x^\star$. In order for us to achieve all these advantages, we introduced the center Lipschitz-condition

$$\|F'(x_0)^{-1} (F'(x) - F'(x_0))\| \leq \omega_0(\|x - x_0\|) \quad \text{for each } x \in \mathcal{D} \tag{1.7}$$

where ω_0 is a strictly increasing and continuous function with the same property as (1.5). Condition (1.7) follows from (1.4) and

$$\omega_0(t) \leq \omega(t) \quad \text{for each } t \geq 0 \tag{1.8}$$

holds in general. Note also that $\frac{\omega(t)}{\omega_0(t)}$ ($t \geq 0$) can be arbitrarily large [6,11,12,7,1,13,14,2,15–18,3]. Using (1.4) one can show

$$\|F'(x)^{-1} F'(x_0)\| \leq \frac{1}{1 - \omega(\|x - x_0\|)} \tag{1.9}$$

for each x in a certain subset $\mathcal{D}_0$ of $\mathcal{D}$ (to be precised later). This estimate leads to majorizing sequence $\{u_n\}$ [1,9,10]. However, using the less expensive and more precise (1.7), we obtain that

$$\|F'(x)^{-1} F'(x_0)\| \leq \frac{1}{1 - \omega_0(\|x - x_0\|)} \quad \text{for each } x \in \mathcal{D}. \tag{1.10}$$