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A compact difference scheme for numerical solutions of second order dual-phase-lagging models of microscale heat transfer

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ABSTRACT

Dual-phase-lagging (DPL) models constitute a family of non-Fourier models of heat conduction that allow for the presence of time lags in the heat flux and the temperature gradient. These lags may need to be considered when modeling microscale heat transfer, and thus DPL models have found application in the last years in a wide range of theoretical and technical heat transfer problems. Consequently, analytical solutions and methods for computing numerical approximations have been proposed for particular DPL models in different settings.

In this work, a compact difference scheme for second order DPL models is developed, providing higher order precision than a previously proposed method. The scheme is shown to be unconditionally stable and convergent, and its accuracy is illustrated with numerical examples.

1. Introduction

Technical advances in nanomaterials and in the applications of ultrafast lasers have lead in last two decades to an increasing interest in non-Fourier models of heat conduction [1–5]. These models try to account for phenomena, such as finite speeds of propagation and wave behaviors, that appear when studying heat transfer at the microscale level, i.e., in very short time intervals or at very small space dimensions [6,7].

The basis for the dual-phase-lagging (DPL) family of models is the introduction of two time lags into the Fourier law [7–9],

$$\mathbf{q}(\mathbf{r}, t+\tau_q) = -k\nabla T(\mathbf{r}, t+\tau_T), \tag{1}$$

where τ_q and τ_T are, respectively, the phase lags in the heat flux vector, $\mathbf{q}$, and the temperature gradient, ∇T, t and $\mathbf{r}$ are the time and spatial coordinates, and k is the conductivity.

When both lags are zero, so that the classical Fourier law is recovered, the combination of (1) with the conservation of energy principle leads to the diffusion or classical heat conduction equation. Otherwise, a partial differential equation with delay is obtained [10,11].

Most commonly, though, first or higher order approximations in the time lags in (1) are used [8,12,13]. In this work, the heat equation resulting from first order approximations,

$$\frac{\partial}{\partial t}T(\mathbf{r}, t) + \tau_q \frac{\partial^2}{\partial t^2}T(\mathbf{r}, t) = \alpha\left(\Delta T(\mathbf{r}, t) + \tau_T \Delta \frac{\partial}{\partial t}T(\mathbf{r}, t)\right), \tag{2}$$

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usually referred to as the DPL model [8], will be denoted DPL(1, 1), and the equations derived from second order approximation in τ_q and up to second order approximation in τ_T will be denoted DPL(2, 1),

$$\frac{\partial}{\partial t}T(\mathbf{r},t)+\tau_q\frac{\partial^2}{\partial t^2}T(\mathbf{r},t)+\frac{\tau_q^2}{2}\frac{\partial^3}{\partial t^3}T(\mathbf{r},t)=\alpha\left(\Delta T(\mathbf{r},t)+\tau_T\Delta\frac{\partial}{\partial t}T(\mathbf{r},t)\right), \tag{3}$$

and DPL(2, 2),

$$\frac{\partial}{\partial t}T(\mathbf{r},t)+\tau_q\frac{\partial^2}{\partial t^2}T(\mathbf{r},t)+\frac{\tau_q^2}{2}\frac{\partial^3}{\partial t^3}T(\mathbf{r},t)=\alpha\left(\Delta T(\mathbf{r},t)+\tau_T\Delta\frac{\partial}{\partial t}T(\mathbf{r},t)+\frac{\tau_T^2}{2}\Delta\frac{\partial^2}{\partial t^2}T(\mathbf{r},t)\right). \tag{4}$$

The construction of numerical solutions for DPL(1, 1) model and variations in different settings has been addressed in previous works (e.g., [14–19]). For DPL(2, 2) models, a Crank–Nicholson type difference scheme was presented in [20]. The objective of this work is to develop a higher order, compact difference scheme for the same problem considered in [20], in a similar way as was done in [17] for DPL(1, 1) models but employing a more direct approach to construct the scheme and prove its stability and convergence.

As in [20], a general equation for heat conduction in one dimension will be considered,

$$\frac{\partial}{\partial t}\left(AT(x,t)+B\frac{\partial}{\partial t}T(x,t)+C\frac{\partial^2}{\partial t^2}T(x,t)\right)=\frac{\partial^2}{\partial x^2}\left(T(x,t)+D\frac{\partial}{\partial t}T(x,t)+E\frac{\partial^2}{\partial t^2}T(x,t)\right), \tag{5}$$

which includes DPL(2, 2), as given by (4), by taking

$$A=\frac{1}{\alpha},\qquad B=\frac{\tau_q}{\alpha},\qquad C=\frac{\tau_q^2}{2\alpha},\qquad D=\tau_T,\qquad E=\frac{\tau_T^2}{2}, \tag{6}$$

and reduces to DPL(2, 1) when $E=0$. The problem is stated for a finite domain $x\in[0,l]$, with Dirichlet boundary conditions,

$$T(0,t)=T(l,t)=0,\quad t\ge 0, \tag{7}$$

and initial conditions

$$T(x,0)=\phi(x),\qquad \frac{\partial}{\partial t}T(x,0)=\varphi(x),\qquad \frac{\partial^2}{\partial t^2}T(x,0)=\psi(x),\quad x\in[0,l]\,. \tag{8}$$

The rest of the paper is organized as follows. In the next section, the new compact difference scheme for DPL(2, 2) model is constructed. In Section 3, after expressing the method as a two-level scheme, the unconditional stability of the method is proved. Next, in Section 4, assuming sufficient regularity of the solution, the consistency of the method is shown, and bounds on the truncation errors are obtained. In the last section, numerical examples are presented, illustrating the higher accuracy of the new method in comparison with the scheme previously proposed in [20].

2. Construction of the compact finite difference scheme

First, two new variables, $v(x,t)$ and $u(x,t)$, are introduced in order to express (5) as a first order system in t,

$$v(x,t)=BT(x,t)+C\frac{\partial}{\partial t}T(x,t)\,, \tag{9}$$

and

$$u(x,t)=AT(x,t)+\frac{\partial}{\partial t}v(x,t)\,. \tag{10}$$

Thus, using (9) and (10), it can be shown that Eq. (5) can be written as

$$\frac{\partial}{\partial t}u(x,t)=a\frac{\partial^2}{\partial x^2}T(x,t)+b\frac{\partial^2}{\partial x^2}v(x,t)+c\frac{\partial^2}{\partial x^2}u(x,t)\,, \tag{11}$$

where

$$a=(C^2+EB^2-BDC-ACE)/C^2,\qquad b=(DC-BE)/C^2,\qquad c=E/C. \tag{12}$$

Consequently, writing (9) and (10) in the form

$$\frac{\partial}{\partial t}T(x,t)=\frac{1}{C}\left(v(x,t)-BT(x,t)\right) \tag{13}$$

and

$$\frac{\partial}{\partial t}v(x,t)=u(x,t)-AT(x,t)\,, \tag{14}$$