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The inverse eigenvalue problem for a Hermitian reflexive matrix and the optimization problem

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ABSTRACT

The inverse eigenvalue problem and the associated optimal approximation problem for Hermitian reflexive matrices with respect to a normal $\{k+1\}$-potent matrix are considered. First, we study the existence of the solutions of the associated inverse eigenvalue problem and present an explicit form for them. Then, when such a solution exists, an expression for the solution to the corresponding optimal approximation problem is obtained.

1. Introduction

In this paper the set of $m \times n$ complex matrices will be denoted by $\mathbb{C}^{m\times n}$ and I_n will stand for the $n \times n$ identity matrix. We will consider the inner product in $\mathbb{C}^{m\times n}$ given by

$$\langle A, B\rangle = \mathrm{trace}\left(B^*A\right), \quad \text{for all } A, B \in \mathbb{C}^{m\times n},$$

where B^* denotes the conjugate transpose of the matrix B. As usual, $\|A\|_F = \sqrt{\langle A, A\rangle}$ stands for the Frobenius norm of A. We recall that a matrix $A \in \mathbb{C}^{n\times n}$ is called reflexive with respect to a certain matrix $J \in \mathbb{C}^{n\times n}$ if $A = JAJ$.

From now on, we will consider a $\{k + 1\}$-potent normal matrix $J \in \mathbb{C}^{n\times n}$ (i.e., $JJ^* = J^*J$ and $J^{k+1} = J,\ k \in \mathbb{N}$). The set of all Hermitian matrices that are reflexive with respect to J will be denoted by $\mathcal{H}J^{n\times n}$, that is,

$$\mathcal{H}J^{n\times n} = \{A \in \mathbb{C}^{n\times n} : A^* = A = JAJ\}.$$

In this paper, we investigate the inverse eigenvalue problem for Hermitian matrices that are reflexive with respect to a $\{k + 1\}$-potent normal matrix J. Specifically, we will solve the following two problems:

Inverse eigenvalue problem: Find all matrices $A \in \mathcal{H}J^{n\times n}$ such that $AX = XD$ for a given matrix $X \in \mathbb{C}^{n\times m}$ and a given diagonal matrix $D \in \mathbb{R}^{m\times m}$.

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In other words, if we solve this inverse eigenvalue problem, we are obtaining all matrices in $\mathcal{HJ}^{n\times n}$ with a prescribed eigenstructure.

Let $\mathcal{S}$ denote the set of all solutions of the previous inverse eigenvalue problem.

Procrustes optimization problem: If $\mathcal{S} \neq \emptyset$, for a given matrix $B \in \mathbb{C}^{n\times n}$, we look for $\hat{A} \in \mathcal{S}$ such that

$$\min_{A\in\mathcal{S}} \|A - B\|_F = \|\hat{A} - B\|_F.$$

In other words, this problem finds the closest matrix $\hat{A}$ (in the set $\mathcal{S}$) to a given matrix B.

The inverse eigenvalue problem has been applied in a wide range of areas such as control theory, mechanic engineering, quantum physics and electromagnetism [1–4]. In the literature the solution of the Procrustes problems have been found for a variety of classes of matrices. For instance, the problem for Hermitian matrices anti-reflexive with respect to a generalized reflection ($J^2 = I_n$ and $J^* = J$) was solved in [5]. The optimization problem related to reflexive matrices with respect to a pair of generalized reflections was studied in [6]. The inverse eigenvalue problem for Hermitian reflexive (anti-reflexive) matrices with respect to a Hermitian tripotent matrix was analyzed in [7]. Also, for structured matrices such as Toeplitz and generalized K-centrohermitian, problems like these ones have been studied in [8,9]. For a left and right inverse eigenvalue problem with reflections we can refer to [10]. The problem treated in this paper extends all these known cases in the literature related to reflexivity.

We recall that for a given matrix $A \in \mathbb{C}^{m\times n}$, its Moore–Penrose inverse is the unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ satisfying $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$. It always exists and it is unique [11]. We will denote $W^{(l)}(A) = I - A^\dagger A$ and $W^{(r)}(A) = I - AA^\dagger$. It is remarkable that $W^{(l)}(A^*) = W^{(r)}(A)$, $W^{(l)}(A)A^\dagger = O$ and $A^*W^{(r)}(A) = O$.

This paper is organized as follows. In Section 2, the structure of the set $\mathcal{HJ}^{n\times n}$ is given. We further analyze necessary and sufficient conditions for the inverse eigenvalue problem to have a solution and an explicit solution is also presented. In Section 3, after analyzing the existence and uniqueness of the Procrustes problem, we find the solution of the optimization problem provided that the set $\mathcal{S}$ is not empty.

2. Inverse eigenvalue problem

Given a matrix $X \in \mathbb{C}^{n\times m}$ and a diagonal matrix $D \in \mathbb{R}^{m\times m}$, we look for solutions of the matrix equation

$$AX = XD \tag{1}$$

satisfying that $A \in \mathbb{C}^{n\times n}$ is Hermitian reflexive with respect to a $\{k+1\}$-potent normal matrix $J \in \mathbb{C}^{n\times n}$.

Notice that the diagonal matrix D has only real entries because A is Hermitian. Since J is normal, it is unitarily diagonalizable. The condition $J^{k+1} = J$ implies that the spectrum of J is included in $\{0\} \cup \Omega_k$, where Ω_k is the set of roots of unity of order k [12]. Then, there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$J = U\,\mathrm{diag}(\omega_1 I_{r_1}, \dots, \omega_t I_{r_t}, O_{r_{t+1}})\,U^*, \tag{2}$$

with $\omega_i \in \Omega_k$, $i = 1, \dots, t$, $r_1 + \cdots + r_t = \mathrm{rank}(J)$ and $r_{t+1} = n - \mathrm{rank}(J)$.

In order to find the structure of the matrix A we partition U^*AU in blocks, of adequate size according to the blocks of the partition of J, as follows:

$$U^*AU = \begin{bmatrix} A_{1,1} & \dots & A_{1,t} & A_{1,t+1} \\ \vdots & \ddots & \vdots & \vdots \\ A_{t,1} & \dots & A_{t,t} & A_{t,t+1} \\ A_{t+1,1} & \dots & A_{t+1,t} & A_{t+1,t+1} \end{bmatrix}. \tag{3}$$

From (2) and (3), the equality $A = JAJ$ yields

$$\begin{cases} A_{t+1,j} = O & \text{for} \quad j \in \{1, \dots, t+1\} \\ A_{j,t+1} = O & \text{for} \quad j \in \{1, \dots, t\} \\ A_{i,j} = \omega_i\omega_j A_{i,j} & \text{for} \quad i, j \in \{1, \dots, t\}. \end{cases}$$

It then follows that $\omega_i\omega_j = 1$ or $A_{i,j} = O$ with $i, j \in \{1, \dots, t\}$. Thus, for each $i, j \in \{1, \dots, t\}$ we get $\omega_i = \overline{\omega}_j$ or the blocks $A_{i,j}$ and $A_{j,i}$ are both zero. We observe that the form of the matrix U^*AU depends on the roots of unity that appear in the decomposition of the matrix J. We will assume that the eigenvalues of the matrix J in (2) are arranged as

$$1, -1, \omega_3, \overline{\omega}_3, \dots, \omega_{p-1}, \overline{\omega}_{p-1}, 0$$

(when 1 and -1 appear). Then, the matrix A has the form

$$A = U\,\mathrm{diag}(A_{1,1}, A_{2,2}, \tilde{A}_{3,4}, \dots, \tilde{A}_{p-1,p}, O)\,U^* \tag{4}$$

where

$$\tilde{A}_{s,s+1} = \begin{bmatrix} O & A_{s,s+1} \\ A_{s+1,s} & O \end{bmatrix} \quad \text{with } s \in \{3, 5, \dots, p-1\} \tag{5}$$

for $p \geq 4$ being an adequate positive integer.