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# Numerical methods for a one-dimensional non-linear Biot's model

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ABSTRACT

Staggered finite difference methods for a one-dimensional Biot's problem are considered. The permeability tensor of the porous medium is assumed to depend on the strain, thus yielding a non-linear model. Some strong two-side estimates for displacements and for pressure are provided and convergence results in the discrete $L^2$-norm are proved. Numerical examples are given to illustrate the good performance of the proposed numerical approach.



## 1. Introduction

Biot's model addresses the time-dependent coupling between the deformation of a porous matrix and the fluid flow inside. The porous matrix is supposed to be saturated by the fluid phase and the flow is governed by Darcy's law. The state of the continuous medium is characterized by the knowledge of the displacements and the fluid pressure at each point of the domain. The one-dimensional theory of isothermal consolidation was first formulated by Terzaghi [1] which was extended to a general 3D consolidation theory by Biot [2,3]. Existence and uniqueness of the solution of the problem are analyzed by Showalter in [4] and by Ženišek in [5]. Nowadays, Biot's models are frequently used in a great variety of disciplines as in geomechanics, petrol engineering, hydrogeology, biomechanics and food processing.

Analytical solution of Biot's model is available only in very special cases, and therefore, numerical methods are commonly used for solving this problem. In general, the solution of complex poromechanics problems is usually approximated by finite elements, see for instance the monograph by Lewis and Schrefler [6]. Problems where the solution is smooth are satisfactorily solved by standard finite element discretizations. Nevertheless, when strong pressure gradients appear, solutions generated by finite element methods exhibit non-physical oscillations. These oscillations can be minimized if stable finite element methods are used. As for Stokes problems, approximation spaces for the vector and the scalar fields satisfying the LBB stability condition [7] can be used. This approach has been analyzed, for example in [8], for the classical quasi-static Biot's model. Nevertheless, these methods still present small oscillations in the pressure approximation when very sharp boundary layers occur.

Naturally, as for finite elements, standard finite-difference schemes may suffer the same unstable behavior in the pressure approximation. In [9], a reason for this non-monotone behavior for one-dimensional consolidation problems has been

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identified, and to avoid this effect, the use of staggered grid discretizations was suggested, theoretically analyzed and tested in two-dimensions [10]. Notice that the use of staggered grids is the way to incorporate a discrete infsup condition in the finite-difference framework, see for example [11]. An extension of this method to the case of discontinuous coefficients through harmonic averaging has been presented in [12]. For other Biot's models, such as the secondary consolidation model [13], the double porosity model [14] and the fully dynamic problem [15], staggered grids have also been successfully applied. In this work, we also apply this technique to non-linear poroelasticity problems in order to avoid the pressure oscillations.

Most of the works in this area treat the linear case. However, the hydraulic permeability of hydrogels and other hydrated soft tissues (e.g., cartilage and intervertebral disc) is deformation dependent [16]. Also, in simulations of hydraulic fracturing of rocks, it is often considered a dependence of the permeability tensor on the stress in exponential form [17]. All these models give rise to non-linear problems. Barucq et al. proved in [18] the existence and uniqueness of the solution of a non-linear fully dynamic poroelastic model, where the nonlinearity appeared in the first equation. Tavakoli and Ferronato [19] used Galerkin's method to prove the existence and uniqueness of solution of the variational problem associated to a non-linear Biot's model where the permeability of the material depended on the strain. The aim of this research is to provide results of stability and convergence of a finite difference scheme on staggered grids for this non-linear Biot's model.

For solving numerically non-linear Biot's models, it is important to develop monotone schemes. Notice that monotone schemes (schemes that satisfy the discrete maximum principle) have remarkable properties providing physically correct solutions, see [20–22]. The study of these schemes will be carried out on a class of one-dimensional problems uncoupled due to the considered boundary conditions. This fact lets us simplify the problem to the case of non-linear parabolic equations with boundary conditions of the second type. For linear parabolic problems, an approach for the construction of second-order monotone finite difference schemes with boundary conditions of the second and third kind without using the differential equation on the boundary of the domain was suggested in [23]. The main idea was based on the extension of the solution of the problem in some small neighborhood of the domain and the use of half-integer grid points. Later, such approach was applied to develop monotone finite difference schemes for non-linear parabolic equations with boundary conditions of the first and third types [24]. In this work, this approach will be extended to the non-linear parabolic problem in the case of boundary conditions of the second type, and as a consequence to the one-dimensional non-linear Biot's model.

The rest of the manuscript is organized as follows. In Section 2, we propose the discretization by finite-difference schemes of non-linear parabolic equations with boundary conditions of the second type. Two-side estimates of the numerical solution and convergence results in the discrete $L^2$-norm are provided in Section 3 and Section 4 respectively. These results will be used to prove the corresponding estimates of the pressure and of the displacements as well as convergence results for the non-linear Biot's model. Numerical results are presented in Section 6, and some conclusions are drawn at the end in Section 7.

## 2. Difference schemes for non-linear parabolic problems with mixed boundary conditions

We consider a finite difference scheme for the solution of the non-linear parabolic differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) = f(x,t), \quad x \in \Omega = (0,l),\ t \in (0,T], \tag{1}$$

with initial and boundary conditions given by

$$\begin{aligned} & u(x,0) = u_0(x), \quad x \in \bar{\Omega}, \\ & u(0,t) = \mu_1(t), \qquad k(u)\frac{\partial u}{\partial x}(l,t) = 0, \quad t \in (0,T]. \end{aligned} \tag{2}$$

We assume that the functions $k(u)$ and $f(x,t)$ are sufficiently smooth in such a way that the solution $u(x,t) \in C^{4,2}(Q_T)$, with $Q_T = [0,l] \times [0,T]$. Moreover, we suppose that there exist values $k_1$ and $k_2$ such that

$$0 < k_1 \le k(u) \le k_2, \quad \forall u \in [m_1, m_2],$$

where $m_1$ and $m_2$ are two constants such that

$$m_1 = \min\left\{\min_{t\in[0,T]} \mu_1(t),\ \min_{x\in[0,l]} u_0(x)\right\} + \int_0^T \min_{x\in[0,l]} f(x,\xi)\,\mathrm{d}\xi,$$

$$m_2 = \max\left\{\max_{t\in[0,T]} \mu_1(t),\ \max_{x\in[0,l]} u_0(x)\right\} + \int_0^T \max_{x\in[0,l]} f(x,\xi)\,\mathrm{d}\xi.$$

Let $N$ and $N_0$ be positive integers and let $h = 2l/(2N+1)$ and $\tau = T/N_0$ be the space and time discretization parameters, respectively. Then, we introduce the uniform grids

$$\bar{\omega}_h = \{x_i = ih,\ i = 0,\ldots,N+1\}, \tag{3}$$

$$\bar{\omega}_t = \{t_n = n\tau,\ n = 0,\ldots,N_0,\ \tau N_0 = T\}. \tag{4}$$