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New results on regularity and errors of harmonic interpolation using Radon projections

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ABSTRACT

We study interpolation of harmonic functions in the unit disk with a finite number of values of the Radon projection along prescribed chords as the input data. We seek the interpolant in the space of harmonic polynomials in such a way that it matches the given projection values exactly. In this setting, we investigate schemes where all chords are divided into two sets of parallel chords. We give necessary and sufficient conditions for a scheme of this type to result in a uniquely solvable interpolation problem. As a second new result, we generalize the previously known error estimates for schemes with equispaced chord angles, both to allow for a larger class of chord choices and to obtain new error estimates in fractional Sobolev norms.

1. Introduction

Multivariate interpolation is known to be significantly more difficult than the univariate case. For instance, while the Lagrangian interpolation problem for polynomials is always uniquely solvable in the univariate case provided that the interpolation nodes are pairwise distinct, this is no longer true in general in the multivariate case. We refer to [1] and the references therein for a survey of multivariate polynomial interpolation. As a further complication, in many practical problems the information about the relevant function comes as a set of functionals which are not point evaluations. For instance, in computer tomography, a table of mean values of a function of d variables on $(d-1)$-dimensional hyperplanes is the data on which the reconstruction is based. Such nondestructive methods have important practical applications in medicine, radiology, geology, etc., and have their theoretical foundation in the work of Johann Radon in the early twentieth century [2].

From the mathematical point of view, the problem is to recover or approximate a multivariate function using information given as integrals of the unknown function over a number of hyperplanes. This topic has been intensively studied since the 1960s using different approaches [3–12] and continues to find many applications. Among the developed reconstruction algorithms are filtered backprojection, iterative reconstruction, direct methods, etc., and some are based on the inverse Radon transform.

In our work, we follow the approach of using direct interpolation by multivariate polynomials. That is, the interpolant is sought in a suitable polynomial space in such a way that it matches the given Radon projections exactly. Methods of this type have been studied, e.g., in [9,13–19].

To improve the approximation accuracy and to reduce the amount of input data required as well as the computational effort, it seems natural to incorporate additional knowledge about the function to be recovered into the approximation

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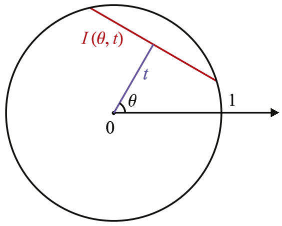

Fig. 1. The chord $I(\theta, t)$ of the unit circle.

method. This was first suggested by Borislav Bojanov. Such problem-specific knowledge is often provided in the form of a partial differential equation which the unknown function satisfies. A recent line of research has focused on the Laplace equation as a model setting for such additional constraints. This means that we seek to interpolate a harmonic function, and thus it is a natural choice to look for interpolants in spaces of harmonic polynomials. Results along these lines are given in [20–24]. The existence of a unique interpolant in the space of harmonic polynomials was shown for a certain family of schemes where all chords are chosen at equal distance to the origin. For the special case of chords forming a regular convex polygon, L_2- and L_∞-error estimates on the unit circle and in the unit disk were proved.

The purpose of this paper is twofold. First, we extend the theory of regularity of schemes, that is, whether a particular choice of chords leads to a harmonic interpolation problem which is always uniquely solvable. Previously, the only class of schemes proved to be regular consisted of chords which had pairwise distinct angles, but the same distance from the origin. Here, we study a different class of schemes consisting of two groups of parallel chords. Within this class, we achieve a complete characterization of regular schemes.

Second, we refine the previously available error analysis, in particular the theory from [23]. Therein, we had to make a particular choice for the constant distance of the chords to the origin in order to obtain an error estimate. In the present paper, we extend the analysis to cover an interval of possible choices of this distance. Furthermore, we analyze the error now more generally in fractional-order Sobolev norms H^s, where the case $H^{1/2}$ may be of particular interest due to the known trace theorems for Sobolev spaces.

The remainder of the paper is structured as follows. In Section 2, we introduce some preliminaries and recall previous results. In Section 3, we describe a new class of regular schemes. In Section 4, we prove new error estimates for chords with equispaced angles. Finally, in Section 5, we give some numerical results.

2. Preliminaries

Let $D \subset \mathbb{R}^2$ denote the open unit disk and ∂D the unit circle. By $I(\theta, t)$, we denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$ from the origin (see Fig. 1). The chord $I(\theta, t)$ is parameterized by

$$\xi \mapsto (t\cos\theta - \xi\sin\theta, t\sin\theta + \xi\cos\theta)^\top, \quad \text{where } \xi \in (-\sqrt{1-t^2}, \sqrt{1-t^2}). \tag{1}$$

Definition 1. Let $f(x, y)$ be a real-valued bivariate function in the unit disk D. The *Radon projection* $\mathcal{R}_\theta(f; t)$ of f in direction θ is defined by the line integral

$$\mathcal{R}_\theta(f; t) := \int_{I(\theta,t)} f(x, y)\, ds = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\cos\theta - \xi\sin\theta, t\sin\theta + \xi\cos\theta)\, d\xi.$$

In this section, we state an interpolation problem for a harmonic function in the unit disk given values of its Radon projections over a set of chords and formulate existence and uniqueness conditions for the corresponding interpolating harmonic polynomial.

If we know *a priori* that the function to be interpolated is harmonic, it is natural to work in the space

$$\mathcal{H}_n = \left\{p \in \Pi_n^2 : \Delta p = 0\right\}$$

of real bivariate harmonic polynomials of total degree at most n, which has dimension $2n + 1$. We prescribe chords

$$\mathcal{I} := \{I_\ell = I(\theta_\ell, t_\ell) : \theta_\ell \in [0, \pi),\ t_\ell \in (-1, 1)\}_{\ell=1}^{2n+1}$$

of the unit circle and associated given values $\Gamma = (\gamma_\ell)_{\ell=1}^{2n+1}$, and wish to find a harmonic polynomial $p \in \mathcal{H}_n$ such that

$$\mathcal{R}_{\theta_\ell}(p, t_\ell) = \int_{I(\theta_\ell,t_\ell)} p(x, y)\, ds = \gamma_\ell, \quad \ell = 1, \ldots, 2n + 1. \tag{2}$$

We use the basis of the harmonic polynomials, with $k \in \mathbb{N}$,

$$\phi_0(x, y) = 1, \qquad \phi_k(x, y) = \mathrm{Re}(x + iy)^k, \qquad \phi_{-k}(x, y) = \mathrm{Im}(x + iy)^k.$$

In polar coordinates, they have the representation

$$\phi_k(r, \theta) = r^k\cos(k\theta), \qquad \phi_{-k}(r, \theta) = r^k\sin(k\theta).$$