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A finite difference moving mesh method based on conservation for moving boundary problems

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ABSTRACT

We propose a velocity-based moving mesh method in which we move the nodes so as to preserve local mass fractions. Consequently, the mesh evolves to be finer where the solution presents rapid changes, naturally providing higher accuracy without the need to add nodes. We use an integral approach which avoids altering the structure of the original equations when incorporating the velocity and allows the solution to be recovered algebraically. We apply our method to a range of one-dimensional moving boundary problems: the porous medium equation, Richards' equation, and the Crank–Gupta problem. We compare our results to exact solutions where possible, or to results obtained from other methods, and find that our approach can be very accurate (1% relative error) with as few as ten or twenty nodes.

1. Introduction

Time-dependent partial differential equations (PDEs) on moving domains, with known fluxes across the boundaries, occur regularly in physical and biological modelling, and must often be solved numerically. The location of the moving boundary is often critical and may require special numerical resolution. In particular, the solution may exhibit singular behaviour at the boundary that is challenging to capture numerically.

Adaptive numerical schemes modify the mesh during the course of computation in response to changes in the dependent variable (or its approximation) in order to achieve greater precision and/or greater efficiency. Generally, an adaptive mesh scheme becomes preferable to a fixed mesh scheme when areas of interest represent only a fraction of the domain being investigated. Increasing the resolution in these areas may then be computationally less expensive than refinement of the mesh over the entire grid. The most common form of mesh adaptivity is h-refinement which involves repeated subdivision of the intervals of a fixed mesh. Other strategies include p-refinement, in which the solution is represented locally by higher order polynomials, and r-refinement in which the mesh points are relocated at each time step. The use of r-refinement has been stimulated by interest in geometric integration, in particular scale invariance (see, e.g., [1]). For scale invariant differential equations, independent and dependent variables are treated alike. An r-refinement method is able to vary the solution and the mesh simultaneously, meaning that the scheme exhibits the same scale invariance as the underlying differential equation. The article by Budd, Huang and Russell [1] and the book by Huang and Russell [2] describe many theoretical and practical aspects of r-adaptivity.

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In this paper a particular r-refinement adaptive scheme is described for the solution of one-dimensional time-dependent PDEs on moving domains. The approach relocates a constant number of nodes by moving the mesh points, keeping a node located at each moving boundary. We show that a mesh with as few as ten or twenty nodes can offer a relative error of less than 1% (see Tables 1–5 in Section 4). The work we present here preserves mass (or relative mass as appropriate), causing the mesh to naturally refine where the solution has high relative density. This is particularly useful for solutions with blow-up, or (as demonstrated here) infinite slope. Attractive aspects of the approach are that no interpolation of the boundary is required, only the moving domain need be discretised, and the continuous movement of the mesh points allows easier inclusion of time integrators.

Under r-refinement nodes may be relocated in many ways, according to the choice of monitor functions [1], and the solution is often found from a moving form of the PDE. A mesh equation is often solved simultaneously with the modified PDE so as to generate the node positions in tandem with the solution, as in the Moving Mesh PDE approach [3,4], the Moving Finite Element method of Miller [5,6], or the parabolic Monge–Ampere approach of Budd and Williams [7,8]. By contrast, in the method described in this paper a single time-dependent equation is solved, that of the mesh, the solution being determined algebraically from a conservation principle. The approach is a finite difference version of the velocity-based moving mesh finite element scheme described by Baines, Hubbard and Jimack in [9,10], in which the mesh equation is based upon conserving a proportion of the total integral (mass) of the dependent variable in the domain. The method in [9,10] differs from methods depending on the technique of equidistribution [3,4,7,8] since equidistribution is not an integral part of the strategy, but is related to the Deformation method of Liao and co-workers [11,12] and to the Geometric Conservation Law (GCL) method of Cao, Huang and Russell [13]. The scheme described herein has been applied to a specific tumour growth problem in [14]. Here we generalise the approach to a wider class of problems, provide key implementation details, and show numerical results for three different nonlinear diffusion problems, each example demonstrating a key feature absent from the problem in [14]. Moreover, we validate our results via comparison with known exact solutions and with results from other (unrelated) approaches.

Throughout we only consider one-dimensional problems. In principle the method can be generalised to higher dimensions, but there are special difficulties with finite differences in higher dimensions and the propensity for mesh tangling is greater. Finite elements are generally considered superior for two- and three-dimensional problems, see [9,10].

The layout of the paper is as follows. In Section 2 we describe the conservation approach, and its finite difference implementation. First, in Section 2.1, we consider mass conserving problems. Then in Section 2.2 these ideas are extended to non mass-conserving problems using a normalisation technique. In Section 3 the schemes are applied to three moving boundary problems, beginning in Section 3.1 with a mass-conserving problem governed by the porous medium equation (PME) (see, e.g., [15]), for which we consider a symmetrical test problem, treated with just one moving boundary. In Section 3.2 the method is applied to a test problem governed by Richards' equation (see [16]). This problem also conserves global mass but the test problem considered is unsymmetrical, so there are two moving boundaries. The third problem, detailed in Section 3.3, is known as the Crank–Gupta or diffusion–absorption problem [17], for which global mass is not conserved. We solve the Crank–Gupta problem for two sets of boundary data, one corresponding to that of the original problem (see [17]), and the other chosen so that we can easily verify our results against a known exact solution. Numerical results for all our examples are provided in Section 4, and some conclusions are presented in Section 5.

We remark finally that our investigation is confined to initial–boundary-value problems for which the solution $u(x, t)$ is one-signed in the interior of the domain, which is necessary for the validity of the method.

2. Conservation-based moving mesh methods

Let $u(x, t)$ be a positive solution of the generic time-dependent scalar PDE

$$\frac{\partial u(x,t)}{\partial t} = \mathcal{L}u(x,t), \quad t > t^0, \ x \in (a(t), b(t)), \tag{1}$$

where $\mathcal{L}$ is a purely spatial differential operator. In all of our examples we have a moving boundary at $x = b(t)$ at which we impose the following Dirichlet and flux boundary conditions

$$u(b(t), t) = 0, \tag{2}$$

$$u(b(t), t)\frac{\mathrm{d}b}{\mathrm{d}t} = 0. \tag{3}$$

The initial condition is

$$u(x, t^0) = u^0(x), \quad x \in (a(t^0), b(t^0)).$$

We introduce a time-dependent space coordinate $\tilde{x}(x, t)$ which coincides instantaneously with the fixed coordinate x. Consider two such coordinates, $\tilde{x}(x_1, t)$ and $\tilde{x}(x_2, t)$, in $(a(t), b(t))$, abbreviated to $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$. The rate of change of the mass in the subinterval $(\tilde{x}_1(t), \tilde{x}_2(t))$ is given by Leibnitz' Integral Rule in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde{x}_1(t)}^{\tilde{x}_2(t)} u(s,t)\,\mathrm{d}s = \int_{\tilde{x}_1(t)}^{\tilde{x}_2(t)} \left(\frac{\partial u(s,t)}{\partial t} + \frac{\partial}{\partial s}(u(s,t)v(s,t))\right)\mathrm{d}s, \tag{4}$$