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The modulus-based nonsmooth Newton's method for solving linear complementarity problems

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ABSTRACT

As applying the nonsmooth Newton's method to the equivalent reformulation of the linear complementarity problem, a modulus-based nonsmooth Newton's method is established and its locally quadratical convergence conditions are presented. In the implementation, local one step convergence is discussed by properly choosing the initial vector and the generalized Jacobian, and a mixed algorithm is given for finding an initial vector. Numerical experiments show that the proposed methods are efficient and accelerate the convergence performance of the modulus-based matrix splitting iteration methods.

1. Introduction

The linear complementarity problem abbreviated as $LCP(M,q)$ consists of finding a vector $z \in \mathbf{R}^n$ such that

$$w = Mz + q \geq 0, \quad z \geq 0, \; z^T w = 0, \tag{1}$$

where $M \in \mathbf{R}^{n\times n}$, and for two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ the order $A \geq B$ means $a_{ij} \geq b_{ij}$ for any i and j.

The $LCP(M,q)$ often arises in many scientific computing and engineering applications, e.g., the linear and quadratic programming, the economies with institutional restrictions upon prices, the optimal stopping in Markov chain, and the free boundary problems; see [1,2] for details.

Many works are devoted to exploit algorithms for solving linear and nonlinear complementarity problems in various situations. One important way for solving (1) is to apply Newton-type methods to solve an equivalent nonsmooth equation, see [3] for a survey. The main feature of these methods is to approximate the nonsmooth (nondifferentiable) problems by a sequence of parameterized smooth (continuously differentiable) problems, and to trace the smooth path which leads to solutions. Chen [4] discussed the local superlinear convergence for nonlinear problems, especially the one step convergence in the linear cases, by a similar technique to those in [5]. For the linear case, from Lemma 3.1 in [5], the initial vector needs to be chosen in

$$B := \begin{cases} \left\{ z \in \mathbf{R}^n : \|z - z^*\| \leq \dfrac{\gamma(z^*)}{\|I - M\|_\infty} \right\} & \text{if } M \neq I \\ \mathbf{R}^n & \text{otherwise,} \end{cases} \tag{2}$$

for one step convergence, where I is the identity matrix, z^* is the solution of (1) and

$$\gamma(z) = \min_{1\leq i\leq n} \{ |z - Mz - q|_i : |z - Mz - q|_i \neq 0 \}.$$

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It is known from [2] that the linear complementarity problem $LCP(M, q)$ is completely equivalent to solving the nonlinear equation

$$F(x) = 0, \tag{3}$$

where F is a function from $\mathbf{R}^n$ into itself defined by

$$F(x) := (M + I)x + (M - I)|x| + q. \tag{4}$$

It is easy to prove that x is a solution of (3) if and only if $z = x + |x|$ is a solution of (1).

By equivalently reformulating (3) as an implicit fixed-point equation, Murty presented a modulus iteration method in [2], which was defined as the solution of a system of linear equations at each iteration. Recently, Bai [6] presented a modulus-based matrix splitting method which not only included the modified modulus method [7] and the nonstationary extrapolated modulus algorithms [8] as its special cases, but also yielded a series of iteration methods, such as modulus-based Jacobi, Gauss–Seidel, SOR and AOR iteration methods, which were extended to more general cases by Li [9]. In addition, Hadjidimos et al. [10] and Zhang [11] proposed scaled extrapolated modulus algorithms and two-step modulus-based matrix splitting iteration methods, respectively. The global convergence conditions are discussed when the system matrix is either a positive definite matrix or an H_+-matrix.

It is well known that Newton's method

$$x^{(i+1)} = x^{(i)} - [G'(x^{(i)})]^{-1}G(x^{(i)}) \tag{5}$$

is a classic method for solving the nonlinear equation

$$G(x) = 0, \tag{6}$$

where $G : \mathbf{R^n} \to \mathbf{R^n}$ is a continuously differentiable function, and hence is a smooth function.

If G is not a smooth function, the formula (5) cannot be used. One method to deal with the nonsmoothness is to split the nonsmooth function into smooth and nonsmooth parts, i.e., G is split as $G = \varphi + \psi$ where φ is continuously differentiable and ψ is continuous nondifferentiable but relatively small. Convergence analysis and error estimation for the Krasnoselskii–Zincenko iteration

$$x^{i+1} = x^i - \varphi'(x^i)^{-1}G(x^i)$$

were given in [12–14]. The Newton-like method:

$$x^{i+1} = x^i - A(x^i)^{-1}G(x^i),$$

where $A(x^i)^{-1}$ is an approximation to $\varphi'(x^i)^{-1}$, is discussed in [15–18].

Furthermore if G is not a smooth function but a locally Lipschitzian function, like (4), let $\partial G(x^{(i)})$ be the generalized Jacobian of G at $x^{(i)}$, where $\partial G(x)$ is the convex hull of all $n \times n$ matrices Z obtained as the limit of a sequence of the form $JG(x_l)$, where $x_l \to x$ and $x_l \notin \Omega_G$, Ω_G is the set of points at which G fails to be differentiable and $JG(x)$ is the Jacobian matrix of partial derivatives whenever x is a point at which the necessary derivatives exist [19]. In this case, instead of (5), Qi and Sun [20] introduced the nonsmooth iterative version

$$x^{(i+1)} = x^{(i)} - V_i^{-1}G(x^{(i)}) \tag{7}$$

where $V_i \in \partial G(x^{(i)})$, for solving (6). A parameterized Newton method is established in [21] with local superlinear convergence results, the generalization of (7).

A lot of research has been done in some structured special matrices, particularly the block matrices, from the practical application; see [22,23] for details. Recently, Aceto and Trigiante [24,25] explored how the advanced matrix results, particularly including the special matrix area, were used in developing other types of numerical methods. Respondek [26,27] presented how to develop an efficient numerical algorithms based on a special block matrix. In this paper, we propose a modulus-based nonsmooth Newton's method from (7) for solving (3), by taking advantage of the technique of the special block matrix. We show that our method is well defined and converges in only one step in application, by properly choosing the generalized Jacobian and the initial vector, under weaker condition than the one in (2).

The rest of the paper is organized as follows. In Section 2 we briefly give some notations, definitions and lemmas which will be used in the paper. In Section 3 we propose a modulus-based nonsmooth Newton's method for solving (1). In Section 4 we give the further convergence analysis and strategies for selecting the generalized Jacobian and the initial vector in the implementation of the proposed method, and we also propose a mixed algorithm. In Section 5 we give some numerical examples to show the efficiency of the proposed algorithms. In the final section we give the concluding remark.

2. Preliminaries

This section is to introduce some notations, preliminary definitions and necessary lemmas.

For $x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$ and a positive integer i, $x^{(i)}$ is devoted to the vector obtained by the ith iteration. We denote $|x| = (|x_1|, |x_2|, \ldots, |x_n|)^T$ and $\mathrm{diag}(x) = \mathrm{diag}\{x_1, x_2, \ldots, x_n\}$, i.e. a diagonal matrix whose main diagonal entries