Journal of Computational and Applied Mathematics 288 (2015) 169–179

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On parameter acceleration methods for saddle point problems

Naimin Zhang

School of Mathematics and Information Science, Wenzhou University, Wenzhou, 325035, PR China

ARTICLE INFO

Article history:
Received 6 June 2014

MSC:
15A06
65F10

Keywords:
Saddle point problems
Parameter acceleration methods
Momentum acceleration
Semi-iterative methods
Convergence

ABSTRACT

For solving saddle point problems, parameter acceleration methods which include Uzawa-type methods are investigated by many researchers in the literature. In this paper, we introduce the inexact Uzawa method with another parameter acceleration, that is, the so-called *momentum acceleration* method for solving saddle point problems. We discuss the convergence conditions of the inexact Uzawa iteration with momentum acceleration and give the optimal momentum factors which minimize the spectral radii of the associated iteration matrices. Numerical results demonstrate the effectiveness of the inexact Uzawa method with momentum acceleration and the mixed parameter acceleration methods for solving saddle point problems.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction

In this paper, we investigate parameter acceleration methods for saddle point problems. Consider the iterative solution of a linear system with the following 2×2 block structure:

$$\mathscr{A}\mathscr{X} \equiv \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \equiv \mathscr{F}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m\times n}$ is full row rank, $f \in \mathbb{R}^n$, $g \in \mathbb{R}^m$, and $m \le n$. We denote the range space and the null space of A by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, the conjugate transpose and transpose of A by A^* and A^T, respectively, and I is an identity matrix of proper order.

Linear systems of the form (1.1) are often called saddle point problems, which arise in many scientific and engineering applications [1], including computational fluid dynamics, constrained optimization, incompressible elasticity, circuit analysis, structured analysis, and so forth. To solve the linear system (1.1), a number of iterative methods, including stationary iterative methods and Krylov subspace iterative methods (often using preconditioners) have been proposed in the literature. In this paper we are concerned with parameter acceleration or relaxation methods, which include Uzawa-type methods [2–10] and SOR-type methods [3,11].

Generally speaking, the above parameter acceleration methods require less arithmetic work per iteration step than Krylov subspace iterative methods, but it is not so easy for parameter acceleration methods to choose optimal parameters in order to minimize the spectral radii of the iteration matrices and obtain the fastest convergence. Uzawa-type methods cover the classical and best known Uzawa method [2], preconditioned Uzawa (PreU) method [7] and parameterized Uzawa (ParU) method [3], where Uzawa method and PreU method belong to single-parameter acceleration methods, and the optimal

E-mail addresses: nmzhang@wzu.edu.cn, nmzhang@aliyun.com.

http://dx.doi.org/10.1016/j.cam.2015.04.028
0377-0427/© 2015 Elsevier B.V. All rights reserved.

parameters of them are well known. ParU method is a double-parameter acceleration method, it can be judged as one of SOR-type methods and is also called generalized successive overrelaxation (GSOR) method. Bai, Parlett and Wang [3] establish the convergence theory for GSOR and obtain the optimal parameters. Chao, Zhang and Lu [12] study the generalized symmetric SOR method (GSSOR) with two parameters and obtain the optimal parameters, and they find that the GSOR method and the GSSOR method have the same optimal convergence factors, that is, the minimal spectral radii of their iteration matrices are equal. Golub, Wu and Yuan [11] propose SOR-like (SORL) methods which belong to single-parameter acceleration methods, and discuss the convergence and the optimal parameters. A complete version for the convergence theory of SORL methods can be also found in [3]. Bai, Golub and Ng [13] propose the Hermitian and skew-Hermitian splitting (HSS) iteration method. Furthermore, by making use of the HSS iteration technique, Bai and Golub [14,15] establish a class of double-parameter accelerated Hermitian and skew-Hermitian splitting (AHSS) iteration methods, and obtain the quasi-optimal parameters which minimize a best-possible upper bound of the corresponding iteration matrix's spectral radius.

In this paper, we introduce another parameter acceleration method, that is, we give the so-called *momentum acceleration* method for solving the linear system (1.1). The idea of momentum acceleration comes from neural network algorithms and it is introduced by Rumelhart, Hinton and Williams [16]. Many researchers have developed the theory about momentum and extended its applications, see, e.g., [17–23]. Here we point out that N. Qian studies its mechanisms. It is shown [19] that in the limit of continuous time, the momentum factor is analogous to the mass of Newtonian particles that move through a viscous medium in a conservative force field, and the behavior of the system near a local minimum is equivalent to a set of coupled and damped harmonic oscillators. The momentum term improves the speed of convergence by bringing some eigencomponents of the system closer to critical damping. And similar results can be obtained for the discrete time case used in computer simulations. Taking the variable momentum parameters, A. Bhaya et al. [17] establish various connections between the conjugate gradient (CG) algorithm and the backpropagation algorithm with momentum acceleration, and they obtain an interesting result that the gradient method with momentum for quadratic functions is a version of the CG method. In [22,23], the author discusses some elementary applications of momentum to numerical optimization and numerical algebra. In this paper, we further make use of momentum to accelerate the convergence for solving saddle point problems.

The rest of this paper is organized as follows. In Section 2 we review some parameter acceleration methods and give different representations for the inexact Uzawa method plus momentum, that is, the inexact Uzawa method plus momentum can be regarded as both a semi-iterative method and a new stationary iteration method. In Section 3 we discuss the convergence conditions of the inexact Uzawa iteration with momentum acceleration for solving saddle point problems, and give the optimal momentum factors which minimize the spectral radii of the associated iteration matrices. Numerical experiments are presented in Section 4. We examine the performance of the momentum acceleration, and compare the different parameter acceleration methods with the preconditioned GMRES method. Finally, some conclusions are drawn in Section 5.

2. Representations of parameter acceleration methods

In this section we briefly review parameter acceleration methods and give their representations. First we give the abstract inexact Uzawa (IU) algorithm [5] as follows:

Algorithm 2.1 (*IU*)**.** Let $Q_1 \in \mathbb{R}^{n\times n}$, $Q_2 \in \mathbb{R}^{m\times m}$ be two nonsingular matrices. Given initial vectors $x^{(0)} \in \mathbb{R}^n$ and $y^{(0)} \in \mathbb{R}^m$, for $k = 0, 1, \ldots,$ compute

$$\begin{cases} x^{(k+1)} = x^{(k)} + Q_1^{-1}(f - Ax^{(k)} - B^T y^{(k)}), \\ y^{(k+1)} = y^{(k)} + Q_2^{-1}(Bx^{(k+1)} + g). \end{cases} \tag{2.1}$$

In Algorithm 2.1, generally Q_1 is an approximate matrix of A and Q_2 is an approximate matrix of the Schur complement matrix $BA^{-1}B^T$.

Denote

$$\mathscr{X}^{(k)} = \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}, \qquad \mathscr{M} = \begin{pmatrix} Q_1 & 0 \\ -B & Q_2 \end{pmatrix}. \tag{2.2}$$

Then the iteration matrix $\mathscr{T}$ of IU is

$$\mathscr{T} = I - \mathscr{M}^{-1}\mathscr{A} = \begin{pmatrix} I - Q_1^{-1}A & -Q_1^{-1}B^T \\ Q_2^{-1}B(I - Q_1^{-1}A) & I - Q_2^{-1}BQ_1^{-1}B^T \end{pmatrix}, \tag{2.3}$$

and (2.1) can be written as

$$\mathscr{X}^{(k+1)} = \mathscr{T}\mathscr{X}^{(k)} + \mathscr{M}^{-1}\mathscr{F}. \tag{2.4}$$

It is known that the iteration (2.4) converges if and only if $\rho(\mathscr{T}) < 1$, where $\rho(\cdot)$ denotes the spectral radius of the corresponding matrix.

With different choices of Q_1 and Q_2, Algorithm 2.1 covers several Uzawa-type algorithms and SOR-type algorithms as follows.

Download English Version:

https://daneshyari.com/en/article/4638400

Download Persian Version:

https://daneshyari.com/article/4638400

Daneshyari.com