Journal of Computational and Applied Mathematics 288 (2015) 180–192

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A modified Tikhonov regularization method[☆]

Xiao-Juan Yang[a,b], Li Wang[a,*]

[a] Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, PR China

[b] School of Mathematical Sciences, Nanjing Normal University, Taizhou College, Taizhou, 225300, PR China

ARTICLE INFO

Article history:
Received 8 January 2014
Received in revised form 28 March 2015

MSC:
65F15

Keywords:
Tikhonov regularization
TSVD
Discrepancy principle
GCV
Arnoldi-based hybrid method

ABSTRACT

Tikhonov regularization and truncated singular value decomposition (TSVD) are two elementary techniques for solving a least squares problem from a linear discrete ill-posed problem. Based on these two techniques, a modified regularization method is proposed, which is applied to an Arnoldi-based hybrid method. Theoretical analysis and numerical examples are presented to illustrate the effectiveness of the method.

1. Introduction

Consider a linear least-squares problem:

$$\min_{x\in\mathbb{R}^n} \|Ax-b\|, \quad A\in\mathbb{R}^{m\times n},\ m\geq n, \tag{1}$$

where and throughout this paper, $\|\cdot\|$ denotes the Euclidean vector norm or the corresponding induced matrix norm. The singular values of the matrix A are assumed of different orders of magnitude close to the origin and some of them may vanish. The minimization problem with a matrix of ill-determined rank is often referred to as a linear discrete ill-posed problem. It may be obtained by discretizing linear ill-posed problems, such as Fredholm integral equations of the first kind with a smooth kernel. This type of integral equations arises in science and engineering when one seeks to determine the cause (the solution) of an observed effect represented by the right-hand side b (the data). Because the entries of b are obtained through observation, they are typically contaminated by a measurement error and also by a discrete error. We denote these errors by $e\in\mathbb{R}^n$ and the unavailable error-free right-hand side associated with b by $\hat{b}\in\mathbb{R}^n$, i.e.,

$$b=\hat{b}+e. \tag{2}$$

We assume that a bound δ for which

$$\|e\|\leq\delta$$

☆ The work is supported by the National Natural Science Foundation of China under grant 10971102, the Natural Science Foundation of Jiangsu Province of China under grant BK2009398.

* Corresponding author.
E-mail addresses: wangli1@njnu.edu.cn, wlsha@163.com (L. Wang).

http://dx.doi.org/10.1016/j.cam.2015.04.011
0377-0427/

is available, and the linear system of equations with the unavailable error-free right-hand side

$$Ax = \hat{b} \tag{3}$$

is consistent. Let $\hat{x}$ denote a desired least-squares solution of (3) in the sense of the minimal Euclidean norm. We seek an approximation to $\hat{x}$ by computing an approximate solution of the available linear system of equations (1). Due to the severe ill-conditioning of A and the error e on the right-hand side b, a solution of (1) typically does not yield a meaningful approximation of $\hat{x}$.

The discrete ill-posed problem (1) of small or moderate size is often solved by the truncated singular value decomposition (TSVD) or Tikhonov regularization, see [1,2] for details.

The basis of these two techniques is the singular value decomposition (SVD) defined as

$$A = U\Sigma V^T, \tag{4}$$

where $U = [u_1, u_2, \ldots, u_m] \in \mathbb{R}^{m\times m}$, $U^TU = I$, $V = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^{n\times n}$, $V^TV = I$ and

$$\Sigma = diag[\sigma_1, \sigma_2, \ldots, \sigma_n].$$

Here $(\cdot)^T$ denotes transposition of $(\cdot)$ and the singular values are ordered as

$$\sigma_1 \geq \sigma_2 \geq \sigma_l > \sigma_{l+1} = \cdots = \sigma_n = 0, \quad l = rank(A).$$

The minimum-norm least-squares solution x_{LS} of (1) is

$$x_{LS} = A^+b = \sum_{j=1}^{l} \frac{u_j^Tb}{\sigma_j} v_j,$$

where $A^+ = \sum_{j=1}^{l} v_j\sigma_j^{-1}u_j^T$ is the Moore–Penrose generalized inverse of A.

By ignoring some small singular values, we get the truncated SVD solution x_k given by

$$x_k = A_k^+b = \sum_{j=1}^{k} \frac{u_j^Tb}{\sigma_j} v_j \tag{5}$$

where $k(1 \leq k \leq l)$ is the truncated parameter and $A_k = \sum_{j=1}^{k} u_j\sigma_j v_j^T$.

We note that $x_k \in span\{v_1, v_2, \ldots, v_k\}$. The singular values σ_j and the coefficients u_j^Tb provide a valuable insight about the properties of the linear discrete ill-posed problem (1); see, e.g., [3,2] for a discussion on applications of the TSVD to the linear discrete ill-posed problems.

Instead of solving (1), Tikhonov regularization solves the minimization problem

$$\min_{x\in\mathbb{R}^n}\{\|Ax - b\|^2 + \mu^2\|Lx\|^2\}, \tag{6}$$

which is commonly referred to as a regularization of the problem (1). The scalar $\mu > 0$ is the regularization parameter, and the matrix $L \in \mathbb{R}^{p\times n}(p \leq n)$ is referred to as the regularization matrix, which is chosen either to be the identity matrix I, or a discrete approximation to a derivation operator. The minimization problem (6) is said to be in *standard form* when $L = I$ and in *general form* otherwise. Many examples of regularization matrices can be found in [4–7].

The matrix L is assumed to satisfy

$$N(A) \cap N(L) = \{0\},$$

where $N(\cdot)$ denotes the null space of $(\cdot)$. Then the Tikhonov minimization problem (6) has a unique solution

$$x_\mu = (A^TA + \mu^2L^TL)^{-1}A^Tb; \tag{7}$$

see, e.g., [1,2] for discussions on Tikhonov regularization.

The regularization parameter can be determined in a variety of ways; see, e.g., [8,1,2,9,10]. In our work, we apply the discrepancy principle [1,2,10] to determine the truncation index k and the regularization parameter μ, so that

$$\|Ax_k - b\| \leq \eta\delta, \tag{8}$$

$$\|Ax_\mu - b\| = \eta\delta, \tag{9}$$

where x_k and x_μ are defined in (5) and (7) respectively, and $\eta \geq 1$ is a user-specified constant independent of δ and is usually fairly close to unity.

Thus the truncation index k satisfies

$$\sum_{j=k+1}^{n}(u_j^Tb)^2 \leq (\eta\delta)^2 \leq \sum_{j=k}^{n}(u_j^Tb)^2.$$

Properties of this method are discussed in, e.g., [1,2].