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A two-grid combined finite element-upwind finite volume method for a nonlinear convection-dominated diffusion reaction equation

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In this paper, we first present a combined finite element-upwind finite volume method for a fully nonlinear convection-dominated diffusion reaction equation and derive its a priori optimal error estimate in H^1-norm and sub-optimal error estimate in L^2-norm for piecewise linear finite element combining with the first order upwind finite volume scheme. Then we study a type of two-grid method for the nonlinear convection-dominated transport equation together with the combined finite element-upwind finite volume method on the fine grid T_h and the streamline diffusion finite element scheme on the coarse grid T_H, which not only significantly reduces the computational cost on nonlinear iterations but also remains the numerical computation stabilized and the approximation accuracy unchanged. A priori error estimate of such designed two-grid method in H^1-norm is proved to be $O(h + H^{\frac{3}{2}})$, showing that the two-grid method achieves the optimal approximation as long as the mesh sizes satisfy $h = O(H^{\frac{3}{2}})$. Finally, a numerical example is carried out to verify the accuracy and efficiency of the present numerical method.

1. Introduction

We consider the following general fully nonlinear convection-dominated diffusion reaction equation

$$\begin{cases} -\nabla \cdot (a(u)\nabla u) + \nabla \cdot (\vec{b}(u)u) = f(u), & x \in \Omega \\ u = 0. & x \in \Gamma \end{cases} \tag{1}$$

in a domain $\Omega \subset R^n$ with boundary Γ. We make the following assumptions for problem (1). The coefficient functions $a(u), \vec{b}(u) = (b_1(u), \ldots, b_n(u))^T, f(u)$ are all bounded and Lipschitz continuous with respect to u, and there exist positive constants A_0, C such that $0 < A_0 \le a(u) < C$, $a(u), b_i(u), f(u) \in C_b^2(\Omega)$, $i = 1, 2, \ldots, n$, where $C_b^2(\Omega)$ is a class of twice continuously differentiable functions in Ω such that all derivatives of the functions up to and including second order are bounded in Ω. In addition, $\nabla \cdot b(u) > 0$, and $|\vec{b}(u)| \gg a(u)$ which reflects the dominant convection effect.

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The investigation of the convection–diffusion problems has been the object of an increasing interest of a number of specialists as well as mathematicians. Special attention is paid particularly to problem with convection dominating over diffusion. It is well known that the straightforward application of the finite element method to singularly perturbed convection–diffusion problems may lead to spurious oscillations in the approximate solution [1]. The reason for this drawback is that the discrete method loses most of the stability properties of the continuous problem. Several methods have been intensively studied to remove such a drawback. A popular idea is to add stabilization terms to the formulation of the problem, such as the Streamline-Upwind/Petrov–Galerkin (SUPG) method [2,3], or streamline diffusion method [4–8], or Galerkin-least-squares method [9–13], however, these schemes need to manually tune the stabilization parameters in order to achieve a good stability. Although there exists some formula to help to find an appropriate stabilization parameter for linear PDE, it is difficult to define such formula for a nonlinear problem. In this sense, upwind finite volume scheme seems more promising because its upwind parameter can be automatically obtained based on the numerical flux.

On the other hand, in many physical problems the conservation of some quantities such as mass or momentum is a basic property of the mathematical model. Hence, it is important to study proper discretization to preserve the same feature, which is not generally guaranteed by the stabilized finite element methods. Upwinding and conservation appear naturally when we are dealing with a finite volume technique.

Based on an upwind modification of a standard Galerkin discretization, the problem of finding a finite element scheme for a convection–diffusion problem whose solution satisfies a discrete mass conservation law was first addressed in [14]. Feistauer [15] proposed and analyzed a semi-implicit scheme for a nonlinear convection–diffusion problem based on a node-centered finite volume approach which prove the convergence of the discrete solution to the exact solution. To maintain the superiority of finite element method on the convergence accuracy and the convenience of dealing with the natural boundary conditions, at the same time, overcome the dominated convection effect and stabilize the numerical computation, in this paper we present a combined finite element-upwind finite volume method [14–18] to solve a fully nonlinear convection-dominated diffusion reaction equation, which discretizes the diffusion and reaction terms with standard finite element scheme. Hashim and Hu [19] applied a similar approach to the semi-linear convection–diffusion equation with a simpler convection term $b(u) \cdot \nabla u$, but none of them uses this method for a fully nonlinear convection-dominated diffusion reaction equation, in which the convection term is in divergence form $\nabla \cdot (b(u)u)$, which is the most general case of convection effect, and the reaction term $f(u)$ is nonlinear too. In this paper we design an appropriate combined finite element-upwind finite volume scheme to solve the aforementioned nonlinear PDE, analyze its a priori error estimate, and obtain the optimal error estimate in H^1 norm and sub-optimal error estimate in L^2 norm for a piecewise linear finite element combining with the first order upwind finite volume scheme, where, to our knowledge, L^2 norm error estimate is established for the present nonlinear model for the first time in this paper.

Besides the stability issue induced by the dominant convection effect, the nonlinear iteration is another important issue when we carry out a numerical simulation for a nonlinear problem on a fine grid, which is always time-consuming and thus expensive. It is well known that two-grid method [20,21] is a discretization technique for efficiently solving nonlinear equations based on two grids with different sizes. This method iteratively approximates the solution of the original nonlinear problem on a coarse grid first, then the symmetry and linear part of the equation are modified on a fine grid. Since the two-grid method only conducts the nonlinear iteration on the coarse grid, and solve a linear equation on the fine grid, the entire computational cost is thus significantly reduced. Two-grid method has been successfully used for solving elliptic boundary value problems [20,21], Stokes equation [22], the steady-state Navier–Stokes equations [23] and other partial differential equations. Wu [24] applied two-grid method combining with mixed finite element method to the reaction–diffusion equations. Chen [25] constructed a two-grid method for the expanded mixed finite element solution of a semilinear reaction–diffusion equations. However, to our knowledge, no one has ever used the two-grid approach in the combined finite element-upwind finite volume method yet.

In this paper, based on two conforming linear finite element spaces V_H defined on the coarse grid with grid size H and V_h defined on the fine grid with grid size h, we consider a two-grid combined finite element-upwind finite volume method for the fully nonlinear convection-dominated diffusion reaction Eq. (1). With the proposed techniques, we solve the original nonlinear problem in the coarse finite element space with a stabilized streamline diffusion-finite element discretization and Picard's linearization, then solve the corresponding linearized problem in the fine finite element space with the combined finite element-upwind finite volume method. Thus, based on the existing convergence result of streamline diffusion finite element scheme [4–8] and the newly obtained convergence theorem of the combined finite element-upwind finite volume method, we are able to attain an optimal error estimate for the present two-grid method. A remarkable fact about this approach is that the coarse grid can be quite coarse and thus the work for solving the nonlinear problem is relatively negligible, in the meanwhile, the approximation accuracy remains the same with that of the numerical method which is adopted on the fine grid.

The rest of this paper is organized as follows. In Section 2, we describe the combined finite element-upwind finite volume method and conduct its a priori error estimate for the nonlinear problem (1). The two-grid combined finite element-upwind finite volume method and its a priori error estimate are given in Section 3. A numerical example is presented to verify the theoretical results in Section 4.

Throughout this paper, the letter C or with its subscript denotes a generic positive constant which does not depend on the mesh sizes and may represent different value at different occurrences.