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Analytically pricing volatility swaps under stochastic volatility

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ABSTRACT

Papers focusing on analytically pricing discretely-sampled volatility swaps are rare in literature, mainly due to the inherent difficulty associated with the nonlinearity in the pay-off function. In this paper, we present a closed-form exact solution for the pricing of discretely-sampled volatility swaps, under the framework of Heston (1993) stochastic volatility model, based on the definition of the so-called average of realized volatility. By working out such a closed-form exact solution for discretely-sampled volatility swaps, this work represents a substantial progress in the field of pricing volatility swaps, as it has: (1) significantly reduced the computational time in obtaining numerical values for the discretely-sampled volatility swaps; (2) improved the computational accuracy of discretely-sampled volatility swaps, comparing with the continuous sampling approximation, especially when the time interval between sampling points is large; (3) enabled all the hedging ratios of a volatility swap to be analytically derived.

1. Introduction

Volatility and variance swaps are essentially forward contracts on annualized realized volatility or variance that provide an easy way for investors to trade future realized volatility or variance against the current implied volatility or variance. The long position of a volatility or variance swap pays a fixed delivery price at expiry and receives the floating amounts of annualized realized volatility or variance, whereas the short position is just the opposite. With the rapid growth of trading variance swaps and volatility swaps in the past two decades, correctly pricing variance swaps and volatility swaps has been an actively pursued research topic; some typical papers include [1–10] etc.

[11] outlined an overview of the current market for volatility derivatives (such as variance and volatility swaps, VIX futures and options) and gave a survey of the literature. However, despite many common features between volatility swaps and variance swaps, the former is viewed to be more difficult to price analytically than the latter because the payoff function involves either square root operator or absolute value operator. As a result, quite a few closed-form solutions have been discovered for the latter (cf. [9]) whereas it is very rare to see a paper discussing closed-form solution for the former. In fact, none has considered the pricing of volatility swaps based on the discretely-sampled realized volatility in the literature.

The main purpose of this paper is to extend [9]'s approach to pricing discretely-sampled volatility swaps. We present a closed-form exact solution for the price of discretely-sampled volatility-average swaps under the Heston stochastic volatility model. There are several reasons that we believe this research will benefit the research community as well as

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market practitioners. Firstly, this study, by working out an exact closed-form solution for the discretely-sampled volatility-average swaps based on the Heston stochastic volatility model, fills a gap that there is no exact pricing formula available for discretely-sampled volatility swaps in the literature. Secondly, this study also demonstrates that our proposed solution approach can be used to work out a lower bound for the standard derivation swap in which the realized volatility is defined as the square root of the average of realized variance. Thirdly, it can be used as a benchmark tool for numerical methods developed to price volatility swaps whose payoff function has made the search of closed-form analytical solution impossible.

During the review of this paper, [12] published a paper, dealing with pricing discretely-sampled variance swaps and other continuous-sampled based variance derivatives using a forward characteristic function (FCF) approach. However, their FCF (their Eq. (23) on Page 149) is of a quite different form from the one we have presented in this paper. While it is quite possible for an FCF to have different analytic forms, it is clear that their FCF contains some minor typos. More importantly, working out a FCF is only the first step in terms of providing a pricing formula for discretely-sampled volatility swaps. [12] only completed this first step, without providing a specific pricing formula for the price of discretely-sampled volatility swaps with the realized volatility being defined as the square root of the average of realized variance as we did in this paper. Therefore, our paper should add a good contribution to this field as not only a correct form of FCF is presented for this type of problems, a pricing formula, verified already through various examples, is also supplied for pricing discretely-sampled volatility swaps with the realized volatility being defined as the square root of the average of realized variance.

The rest of this paper is organized into four sections. For the easiness of reference, we shall start with a description of our solution approach and our analytical formula for the volatility swaps in Section 2. Then, some numerical examples are given in Section 3, demonstrating the correctness of our solution from various aspects. In the mean time, we also provide some comparisons to continuous sampling models and discussions on other properties of the volatility swaps. Our conclusions are stated in Section 4.

2. Our solution approach

In this section, we use the [13] stochastic volatility model to describe the dynamics of the underlying asset. We then present our pricing approach to price discretely-sampled volatility-average swaps and obtain a closed-form exact solution.

2.1. Volatility swaps

A volatility swap is a forward contract on realized historical volatility of the specified underlying equity index. The amount paid at expiration is based on a notional amount times the difference between the realized volatility and implied volatility. More specifically, assuming the current time is 0, the value of a volatility swap at expiry can be written as $(RV(0, N, T) - K_{vol}) \times L$, where the $RV(0, N, T)$ is the annualized realized volatility over the contract life $[0, T]$, K_{vol} is the annualized delivery price for the volatility swap, which is set to make the value of a volatility swap equal to zero for both long and short positions at the time the contract is initially entered. To a certain extent, it reflects market's expectation of the realized volatility in the future. L is the notional amount of the swap in dollars per annualized volatility point squared. The realized volatility is always discretely sampled over a time period $[0, T]$, with T being referred to as the *total* sampling period, in comparison with the sampling period that is used to define the time span between two sampling points within the total sampling period.

At the beginning of a contract, it is clearly specified the details of how the realized volatility, $RV(0, N, T)$, should be calculated. Important factors contributing to the calculation of the realized volatility include underlying asset(s), the observation frequency of the price of the underlying asset(s), the annualization factor, the contract lifetime, the method of calculating the volatility. As illustrated in [4,14], there are at least two different measures of realized volatility:

$$RV_{d1}(0, N, T) = \sqrt{\frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2} \times 100 \tag{1}$$

or

$$RV_{d2}(0, N, T) = \sqrt{\frac{\pi}{2NT}} \sum_{i=1}^{N} \left| \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right| \times 100 \tag{2}$$

where $t_i, i = 0 \ldots N$, is the ith observation time of the realized volatility in the pre-specified time period $[0, T]$, and $t_0 = 0, t_N = T$. S_{t_i} is the closing price of the underlying asset at the ith observation time t_i, and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. For most of the traded variance swaps, or even over-the-counter ones, the sampling period is usually constant to make the calculation of the realized volatility easier. Therefore, we assume equally-spaced discrete observations in the period $[0, T]$ in this paper. As a result, the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

Although both of these two definitions can be used to measure the realized volatility, they are slightly different. The definition $RV_{d1}(0, N, T)$ is essentially calculated as the square root of average realized variance. [4] termed a volatility swap contract using this measurement to calculate realized volatility as a standard derivation swap. On the other hand,