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Extended convergence results for the Newton–Kantorovich iteration

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ABSTRACT

We present new semilocal and local convergence results for the Newton–Kantorovich method. These new results extend the applicability of the Newton–Kantorovich method on approximate zeros by improving the convergence domain and ratio given in earlier studies by Argyros (2003), Cianciaruso (2007), Smale (1986) and Wang (1999). These advantages are also obtained under the same computational cost. Numerical examples where the old sufficient convergence criteria are not satisfied but the new convergence criteria are satisfied are also presented in this study.

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Let $U(x_0, R)$ stand for the open ball centered at $x_0 \in \mathcal{X}$ and of radius $R > 0$ and let $\bar{U}(x_0, R)$ stand for its closure. We shall also denote by $Ł(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$.

In this study we are concerned with the problem of approximating a locally unique zero $x^\star$ of F, where F is a Fréchet-differentiable operator defined on $\bar{U}(x_0, R)$ and with values in $\mathcal{Y}$. Many problems are reduced to finding zeros of operators using Mathematical Modeling [1–3]. The zeros of these operators can be found in closed form only in special cases. That is why most solution methods for these problems are iterative. In Computational Sciences the practice of Numerical Functional Analysis is essentially connected to variants of Newton's method [4–6,1,7–9,2,10–15,3,16,17].

The Newton–Kantorovich method defined by

$$x_n = x_{n-1} - F'(x_{n-1})^{-1}F(x_{n-1}) \quad \text{for } x_0 \in \mathcal{X} \text{ and each } n \in \mathbb{N} \tag{1.1}$$

is undoubtedly the most popular method for generating a sequence $\{x_n\}$ approximating the solution $x^\star$. The convergence analysis of iterative methods is usually divided into two categories: semilocal and local convergence analysis. In the semilocal convergence analysis one derives convergence criteria from the information around an initial point whereas in the local analysis one finds estimates of the radii of convergence balls from the information around a solution. There is a plethora of local as well as semilocal convergence results for Newton's method defined above. We refer the reader to [4–6,1,7–9,2,10–15,3,16,17] and the references therein. The celebrated Kantorovich theorem is an important tool in numerical analysis,

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e.g. for providing sufficient criteria for the convergence of Newton's method to zeros of polynomials or of systems of non-linear equations. This theorem is also important in Nonlinear Functional Analysis, where it is also used as a semilocal result for establishing the existence of a solution of a nonlinear equation in an abstract space.

In the present study we are being motivated by the work of Cianciaruso [10] on approximate zeros for the Newton–Kantorovich method and optimization considerations. We show how to extend the applicability of these results under the same computational cost.

The paper is organized as follows: In Section 2 we introduce some definitions and state the earlier results as well as the results of this paper. The semilocal and local analyses are presented in Sections 3 and 4, respectively. Numerical examples are presented in the concluding Section 5.

2. Preliminaries

It is well known that if the initial point x_0 is close enough to the solution x^*, then sequence $\{x_n\}$ is ultrafast convergent to x^* [1,2,12]. The ultrafast convergence of sequences $\{x_n\}$ is related to the definition of approximate zero introduced by Smale in [3].

Definition 2.1. A point x_0 is said to be an approximate-type zero of F if $\{x_n\}$ is well defined and there exist A_0 and A such that $0 < A_0 < 1$, $0 < A < 1$ and

$$\|x_{n+1} - x_n\| \leq A_0 (A_0 A)^{2^{n-1}-1} \|x_1 - x_0\| \quad \text{for each } n \in \mathbb{N}. \tag{2.1}$$

In the literature they use $A_0 = A = 1/2$ (see e.g. [10,3]). In this study A_0 can be smaller than A (see the proof of Theorem 3.2 and Remark 3.3) which leads to more precise estimates on $\|x_{n+1} - x_n\|$.

Clearly, if x_0 is an approximate-type zero for F, then the sequence $\{x_n\}$ is convergent and its limit point x^* is a zero of F, $F(x^*) = 0$. The corresponding definitions by Smale in [3] and Cianciaruso in [10] are

$$\|x_{n+1} - x_n\| \leq \left(\frac{1}{2}\right)^{2^n-1} \|x_1 - x_0\| \quad \text{for each } n \in \mathbb{N} \tag{2.2}$$

and

$$\|x_{n+1} - x_n\| \leq A^{2^n-1} \|x_1 - x_0\| \quad \text{for each } n \in \mathbb{N} \tag{2.3}$$

respectively. Notice that the new error estimates can be smaller than the old ones for sufficiently small A_0 and A.

Let $F : \bar{U}(x_0, R) \to \mathcal{Y}$ be analytic and $F'(x_0)^{-1} \in Ł(\mathcal{X}, \mathcal{Y})$. Then, Smale in [3] defined

$$\gamma = \gamma(x_0) = \sup_{n>1} \left\| \frac{F'(x_0)^{-1} F^{(n)}(x_0)}{n!} \right\|^{\frac{1}{n-1}}, \tag{2.4}$$

$$\eta = \eta(x_0) = \|F'(x_0)^{-1} F(x_0)\| \tag{2.5}$$

and

$$\alpha = \gamma \eta, \tag{2.6}$$

where $F^{(n)}$ stands for the nth Fréchet-derivative of operator F.

Smale proved that if

$$\alpha < 0.130707, \tag{2.7}$$

then x_0 is an approximate zero of F. This result does not hold if F is not analytic on $\mathcal{X}$. Later, Rheinboldt in [15] proved that if $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$, where $D \subset \mathcal{X}$ is open and

$$\alpha < 0.11909, \tag{2.8}$$

then, sequence $\{x_n\}$ converges. Smale's proof was based on the Newton–Kantorovich theorem [12].

Theorem 2.2. *Suppose: $F : \bar{U}(x_0, R) \to \mathcal{Y}$ is Fréchet-differentiable on $U(x_0, R)$ and F' is Lipschitz continuous; $F'(x_0)^{-1} \in Ł(\mathcal{Y}, \mathcal{X})$. Set*

$$l = \sup_{x \neq y} \frac{\|F'(x_0)^{-1}(F'(x) - F'(y))\|}{\|x - y\|},$$

$$h = l\eta, \qquad t^* = \frac{2\eta}{1 + \sqrt{1 - 2h}},$$

$$t_0 = 0, \qquad t_1 = \eta,$$

$$t_{n+1} = t_n + \frac{l(t_n - t_{n-1})^2}{2(1 - l t_n)} \quad \text{for each } n \in \mathbb{N}$$

where η is defined in (2.5).