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A finite element method for the buckling problem of simply supported Kirchhoff plates

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ABSTRACT

The aim of this paper is to develop a finite element method to approximate the buckling problem of *simply supported* Kirchhoff plates subjected to general plane stress tensor. We introduce an auxiliary variable $w := \Delta u$ (with u representing the displacement of the plate) to write a variational formulation of the spectral problem. We propose a conforming discretization of the problem, where the unknowns are approximated by piecewise linear and continuous finite elements. We show that the resulting scheme provides a correct approximation of the spectrum and prove optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. Finally, we present some numerical experiments supporting our theoretical results.

1. Introduction

The finite element method for the approximation of eigenvalue problems is the object of great interest from both the practical and theoretical point of view. We refer to [1–3] and the references therein for the state of art in this subject area. This paper deals with the analysis of the elastic stability of plates, in particular the so-called *buckling problem*. This problem has attracted much interest since it is frequently encountered in engineering applications such as bridge, ship, and aircraft design. It can be formulated as a spectral problem whose solution is related with the limit of elastic stability of the plate (i.e., eigenvalues-buckling coefficients and eigenfunctions-buckling modes).

The buckling problem has been studied for years by many researchers, being the Kirchhoff–Love and the Reissner–Mindlin plate theories the most used. For the Reissner–Mindlin theory, in [4] was performed the analysis of the buckling problem of a clamped plate modeled by the Reissner–Mindlin equations. On the other hand, in [5,3,6] different formulations for the buckling problem for a thin plate subjected to clamped boundary conditions and modeled by the Kirchhoff–Love theory are considered, while [7] deals with non-conforming methods for the vibration and buckling problems of the biharmonic equation with general boundary conditions.

One of the most well-known mixed methods to deal with the source problem of thin plates modeled by the biharmonic equation is the method introduced by Ciarlet and Raviart [8]. This was thoroughly studied by many authors (see, for

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instance, [9,10], [11, Section 3(a)], [12, Section 4(a)], [13, Section III.3], [14,15]). The method was applied to the plate vibration problem in [1, Section 11.3], [16] and [3, Section 7(b)], where it was proved to converge for finite elements of degree $k \geq 2$; moreover, for this problem a procedure for accelerating the convergence of the approximation has been studied in [17,18] for finite elements of degree $k \geq 2$ and $k = 1$, respectively. A formulation of the eigenvalue problem for the Stokes equation, which turns out to be equivalent to a plate buckling problem, is also analyzed in [3, Section 7(d)], where it is proved to converge for degree $k \geq 2$, as well.

We observe that the buckling problem of a simply supported and uniformly compressed Kirchhoff plate is simpler than the case when a general plane stress tensor is applied. In fact, the solution of the problem can be related with the solution of the Laplace eigenvalue problem with homogeneous boundary conditions. We also note that the same happens for the vibration problem of a simply supported Kirchhoff plate. However, this is not true in the case when the plate is subjected to general plane stress tensor, which is the case that we will study in this work.

Conforming finite element methods for the primal formulation of the biharmonic equation involve C^1 finite elements, which are quite complicated even in two dimensions. An alternative is to use classical non-conforming finite elements as was studied in [7] for the vibration and buckling problems. The aim of this paper is to analyze a conforming discretization based on piecewise linear and continuous finite element of a variational formulation of the buckling problem of simply supported Kirchhoff plates and subjected to general plane stress tensor.

The method is based on the idea introduced by Ciarlet and Raviart [8], and consists in the introduction of an auxiliary variable $w := \Delta u$ (with u being the transverse displacement of the mean surface of the plate) to write a variational formulation of the spectral problem. To analyze the continuous problem, we introduce the so-called solution operator (whose eigenvalues are the reciprocals of the buckling coefficients) which is a compact operator. We propose a conforming discretization based on piecewise linear and continuous finite elements for the two variables. We use the so-called Babuška–Osborn abstract spectral approximation theory (see [1]) to show that the resulting scheme provides a correct approximation of the spectrum and prove optimal order error estimates for the eigenfunctions and a double order for the eigenvalues.

The outline of the paper is as follows: we introduce in Section 2 the variational formulation of the buckling eigenvalue problem, define a solution operator and establish its spectral characterization. In Section 3, we introduce the finite element discrete formulation and describe the spectrum of a discrete solution operator. In Section 4, we prove that the numerical scheme provides a correct spectral approximation and establish optimal order approximation of the eigenfunctions. We end this section by proving that an improved order of convergence holds for the approximation of the eigenvalues. In Section 5, we report some numerical tests which confirm the theoretical order of the error and allow us to assess the performance of the proposed method. Finally, we summarize some conclusions in Section 6.

Throughout the article we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote with c and C, with or without subscripts, tildes or hats, generic constants independent of the mesh parameter h, which may take different values in different occurrences. Moreover $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support contained in Ω. Finally, we will use the following notation for any 2×2 tensor field $\boldsymbol{\tau}$, any 2D vector field $\boldsymbol{v}$, and any scalar field v:

$$\operatorname{div} \boldsymbol{v} := \partial_1 v_1 + \partial_2 v_2, \qquad \nabla v := \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix},$$

$$\mathbf{div}\, \boldsymbol{\tau} := \begin{pmatrix} \partial_1 \tau_{11} + \partial_2 \tau_{12} \\ \partial_1 \tau_{21} + \partial_2 \tau_{22} \end{pmatrix}.$$

Moreover, we denote

$$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. The spectral problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with polygonal boundary occupied by the mean surface of a plate, simply supported on its whole boundary Γ (see [19,20]). The plate is assumed to be homogeneous, isotropic, linearly elastic, and sufficiently thin as to be modeled by Kirchhoff–Love equations. We denote by u the transverse displacement of the mean surface of the plate.

The buckling problem of a plate, which is subjected to a plane stress tensor field $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^{2\times 2}$, $\boldsymbol{\sigma} \neq 0$ reads as the following eigenvalue problem:

$$\begin{cases} \Delta^2 u = -\lambda \operatorname{div} (\boldsymbol{\sigma} \nabla u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where in this case λ is the critical load. To simplify the notation we have taken the Young modulus and the density of the plate, both equal to 1. The applied stress tensor field is assumed to satisfy the equilibrium equations:

$$\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{\sigma} \quad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{div}\, \boldsymbol{\sigma} = 0 \quad \text{in } \Omega. \tag{2.3}$$