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Exponential mean square stability of the theta approximations for neutral stochastic differential delay equations

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ABSTRACT

In this paper, a split-step theta (SST) method is introduced and analyzed for neutral stochastic differential delay equations (NSDDEs). It is proved that the SST method with $\theta \in [0, 1/2]$ can recover the exponential mean square stability of the exact solution with some restrictive conditions on stepsize and the drift coefficient, but for $\theta \in (1/2, 1]$, the SST can reproduce the exponential mean square stability unconditionally. Then, based on the stability results of SST scheme, we examine the exponential mean square stability of the stochastic linear theta (SLT) approximation for NSDDEs and obtain the similar stability results to that of the SST method. Moreover, for sufficiently small stepsize, we show that the decay rate as measured by the Lyapunov exponent can be reproduced arbitrarily accurately. These results show that different values of theta will drastically affect the exponential mean square stability of the two classes of theta approximations for NSDDEs.

1. Introduction

In practice, many system models are described by differential delay equations of neutral type. The models involve not only time delays in the state but also in the state derivatives (see [1–5]). Recently, increasing efforts have been devoted to cope with its stochastic version, known as neutral stochastic differential delay equations (NSDDEs) (see [6–15]). Since most of these equations cannot be solved explicitly, numerical approximations became to be an important tool in studying stochastic systems of neutral type (see [15–20]). Recently, mean square stability theory for numerical simulations of stochastic systems, including stochastic ordinary differential equations (SODEs), stochastic differential delay equations (SDDEs) and NSDDEs, has raised widespread concerns.

Stochastic linear theta (SLT) method is the simplest and most often used method. For linear SODE, the mean square stability of the SLT was investigated in [21–25]. These stability results clearly show that for $\theta \in [0, 1/2)$, the SLT method can preserve the exponential mean square stability with a stepsize restriction, and for $\theta \in [1/2, 1]$, the SLT method can reproduce the stability unconditionally. For nonlinear SODEs, Higham et al. [26] revealed that the backward Euler–Maruyama (BEM) ($\theta = 1$) can reproduce the exponential mean square stability of the SODEs with global Lipschitz condition for sufficiently small stepsize. Under the non-global Lipschitz condition, our previous work [25] examined the theta's effects on the exponential mean square stability and revealed that the linear growth condition on the drift coefficient is necessary for the

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SLT with $\theta \in [0, 1/2)$ to be mean square stable, but for $\theta \in (1/2, 1]$, the SLT can reproduce the exponential mean square stability without the linear growth condition. For SDDEs, Mao [27] investigated the exponential mean square stability of the Euler–Maruyama (EM) method (SLT with $\theta = 0$) under the global Lipschitz condition. Our previous work [28] proved that the SLT method can inherit the exponential mean square stability of the exact solution for SODEs and SDDEs, and the decay rate of the SLT solution approach that for the exact solution as the stepsize tends to zero. However, it is still unknown about the exponential stability and decay rate of SLT method for NSDDEs.

Recently, our co-author Huang [29] proposed a split-step theta (SST) method to approximate the SODEs and extended it to solve SDDEs in [30]. For the special case $\theta = 0$, this approximation is actually EM approximation, and for the case $\theta = 1$, this approximation is equivalent to the split-step backward Euler (SSBE) method, which was investigated for SODEs in [26,31] and was applied to SDDEs in [32]. Higham et al. [26] showed that SSBE method can reproduce the exponential mean square stability of SODEs under the global Lipschitz condition. Wang and Gan [32] proved the positive stability result for the SSBE method of SDDEs without the global Lipschitz condition. Our previous work [29,30] revealed that the SST method with $\theta \in (1/2, 1]$ can share the exponential mean square stability of the exact solution to SODEs or SDDEs unconditionally, nevertheless, for $\theta \in [0, 1/2]$, the linear growth condition would be needed for the SST method to be mean square stable. Our work [28] also examined the stability of SST method and obtained the similar stability results to that of the SLT method. Due to the difficulty arising from the neutral term, split-step method has not been used to approximate the NSDDEs.

This paper aims to examine the exponential mean square stability of the two classes of theta approximations (SST and SLT) for NSDDEs. The important contributions of the current paper can now be highlighted as follows:

- This paper shows that both the two theta approximations can not only reproduce the exponential mean square stability, but also preserve the bound of Lyapunov exponent for sufficient small stepsize, which may measure the decay rate of the numerical solutions.
- Our main results are based on a coupled monotone condition, which implies that the drift and diffusion coefficients may be non-global Lipschitz continuous.

The rest of the paper is organized as follows. Section 2 begins with notations and preliminaries, then introduces the SST and SLT schemes for NSDDEs. Section 3 examines the conditions under which the SST scheme can reproduce the exponential mean square stability of the exact solution. Based on the stability results in Section 3, Section 4 devotes to investigating the exponential mean square stability of the SLT scheme for NSDDEs. Section 5 gives the numerical experiments to confirm the theoretical results.

2. Notations and preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. If A is a vector or matrix, its transpose is denoted by A^T. Let $|\cdot|$ denote both the Euclidean norm in $\mathbb{R}^n$ and the trace (or Frobenius) norm in $\mathbb{R}^{n\times d}$ (denoted by $|A| = \sqrt{\mathrm{trace}(A^TA)}$). $a \vee b$ represents $\max\{a, b\}$ and $a \wedge b$ denotes $\min\{a, b\}$. Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathfrak{F}_0$ contains all P-null sets. Let $w(t)$ be a d-dimensional Brownian motion defined on this probability space.

Let $N : \mathbb{R}^n \mapsto \mathbb{R}^n, f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n, g : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$ be Borel measurable functions. Let us consider the n-dimensional NSDDE of the form

$$d[x(t) - N(x(t-\tau))] = f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dw(t), \quad t > 0 \tag{2.1}$$

with initial data $x(t) = \varphi(t) \in C([-\tau, 0]; \mathbb{R}^n)$, where the delay $\tau > 0$ is a fixed constant. For the purpose of stability, assume that $N(0) = f(0, 0) = 0, g(0, 0) = 0$. This shows that (2.1) admits a trivial solution. We also assume that the neutral term N satisfies the contractive property and f and g satisfy the local Lipschitz condition:

Assumption 2.1 (*Contractive Mapping*)**.** There exists a positive constant $\kappa \in (0, 1)$ such that

$$|N(x) - N(y)| \leq \kappa|x - y| \tag{2.2}$$

for all $x, y \in \mathbb{R}^n$.

Assumption 2.2 (*Local Lipschitz Condition*)**.** f and g satisfy the local Lipschitz condition, that is, for each $j > 0$ there exists a positive constant K_j such that for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\bar{y}| \vee |\bar{y}| \leq j$,

$$|f(x, y) - f(\bar{x}, \bar{y})| \vee |g(x, y) - g(\bar{x}, \bar{y})| \leq K_j(|x - \bar{x}| + |y - \bar{y}|). \tag{2.3}$$

The local Lipschitz condition together with a monotone condition (for example, condition (2.4)) allows the SDDE (2.1) to admit a global solution (see [33]). Before investigating the numerical stability, we firstly give the stability criterion of NSDDE (2.1), which is the special case of Corollary 12 in [33]: