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An improved algorithm for the evaluation of Cauchy principal value integrals of oscillatory functions and its application[☆]

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ABSTRACT

A new interpolatory-type quadrature rule is proposed for the numerical evaluation of Cauchy principal value integrals of oscillatory kind $⨍_{-1}^{1} \frac{f(x)}{x-\tau} e^{i\omega x} dx$, where $\tau \in (-1, 1)$. The method is based on an interpolatory procedure at Clenshaw–Curtis points and the singular point, and the fast computation of the modified moments with Cauchy type singularity. Based on this result, a new method is presented for the computation of the oscillatory integrals with logarithmic singularities too. These methods enjoy fast implementation and high accuracy. Convergence rates on ω are also provided. Numerical examples support the theoretical analyses.

1. Introduction

We consider the problem of numerical evaluation of the Cauchy principal value integrals of oscillatory functions

$$I_\omega(f;\tau) := ⨍_{-1}^{1} \frac{f(x)}{x-\tau} e^{i\omega x} dx = \lim_{\epsilon\to 0^+} \int_{|x-\tau|\ge\epsilon} \frac{f(x)}{x-\tau} e^{i\omega x} dx, \quad -1<\tau<1, \tag{1.1}$$

where f is assumed to be a smooth function, $i^2 = -1$ and $\omega \in \mathbb{R}$. The Cauchy principal value integral is a well known extension of the improper integral which can be found in many analysis textbooks such as [1]. The integral (1.1) exists if the function f satisfies Hölder's condition on $[-1, 1]$. When $|\omega|$ is large, the integrand is highly oscillatory, which will lead to serious difficulties additionally.

Some methods have been presented on this issue. Okecha [2] proposed two kinds of quadrature rules by replacing the function f with the corresponding Lagrange interpolation polynomial based on the zeros of Legendre polynomial and the singular point τ (or not). An alternative method was proposed by Okecha [3] using a special case of Hermite interpolation (Taylor series at τ). A new numerically stable algorithm was proposed by Capobianco and Criscuolo [4] based on an interpolation procedure at the zeros of the orthogonal polynomials with respect to Jacobi weight. Wang and Xiang [5] presented an efficient method, which shares the uniform convergence, by approximating f with the interpolation polynomial $p_{N-1}(x) =$

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$\sum_{j=0}^{N-1''} c_j T_j(x)$ of degree $N-1$ at the N Clenshaw–Curtis points $\{\cos(\frac{j\pi}{N-1})\}_{j=0}^{N-1}$ in terms of Chebyshev polynomials of the first kind, where the double prime denotes a sum whose first and last terms are halved. Then, it yields a quadrature rule for (1.1)

$$I_\omega(f;\tau) \simeq \fint_{-1}^{1} \frac{p_{N-1}(x)}{x-\tau} e^{i\omega x} dx = \int_{-1}^{1} \frac{p_{N-1}(x)-p_{N-1}(\tau)}{x-\tau} e^{i\omega x} dx + p_{N-1}(\tau) \fint \frac{e^{i\omega x}}{x-\tau} dx. \tag{1.2}$$

The first integral of the right hand side of (1.2) can be analytically computed as follows [5]:

$$\int_{-1}^{1} \frac{p_{N-1}(x)-p_{N-1}(\tau)}{x-\tau} e^{i\omega x} dx = 2 \sum_{k=0}^{N-2}{}' T_k(\tau) \sum_{j=0}^{N-k-2} c_{j+k+1} M_j, \tag{1.3}$$

where the prime denotes a sum whose first term is halved, $M_j = \int_{-1}^{1} U_j(x) e^{i\omega x} dx$, and $U_j(x)$ is the Chebyshev polynomial of the second kind of degree j (for more details see [5]). The second integral can be computed in the following closed form [4,2]:

$$\fint_{-1}^{1} \frac{e^{i\omega x}}{x-\tau} dx = e^{i\omega\tau} \left[\mathrm{Ci}(u_1) - \mathrm{Ci}(|u_2|)\right] + i e^{i\omega\tau} \left[\mathrm{Si}(u_1) + \mathrm{Si}(|u_2|)\right], \tag{1.4}$$

where $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$ are the cosine and sine integrals defined in [6, p. 231], $u_1 = \omega(1-\tau)$ and $u_2 = -\omega(1+\tau)$. However, from Eq. (1.3), it can be seen that the computational complexity of this method is $O(N^2)$.

Later, Keller in [7] proposed a new automatic quadrature following the same idea, but the implementation is different and only needs $O(N\log(N))$ operations. He constructed an indefinite integral of oscillatory and singular functions based on a five-term recurrence relation (see Eq. (2.8) of [7]). Furthermore, a new closed form for the second integral on the right hand side of (1.2) was also given in [7]

$$\fint_{-1}^{1} \frac{e^{i\omega x}}{x-\tau} dx = e^{i\omega\tau} \left[\mathrm{sgn}(\omega)\pi i + \mathrm{E}_1(i\omega(1+\tau)) - \mathrm{E}_1(i\omega(\tau-1))\right], \tag{1.5}$$

where $\mathrm{E}_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$ is the exponential integral, and $\mathrm{sgn}(x)$ is the sign function.

Recently, another implementation for (1.2) appeared in Wang et al. [8]. The detailed implementation was listed in Algorithm 2 of [8] and the computation was based on a three-term recurrence relation and the total complexity is also $O(N\log(N))$. It is worth mentioning that Wang and Xiang in [9] presented another method which is very simple and fast if f is analytic in a sufficiently large complex region containing $[-1, 1]$.

Another strategy for this problem is that the original integral is separated into two oscillatory integrals

$$\fint_{-1}^{1} \frac{f(x)}{x-\tau} e^{i\omega x} dx = \int_{-1}^{1} \frac{f(x)-f(\tau)}{x-\tau} e^{i\omega x} dx + f(\tau) \fint_{-1}^{1} \frac{e^{i\omega x}}{x-\tau} dx. \tag{1.6}$$

The second integral of the right side of (1.6) can be computed by the above closed form. For the first integral of the right side of (1.6), an improved method based on [3] was proposed by Chen [10], and Filon-type-Clenshaw–Curtis method is applied in [11]. Li et al. [12] calculated it by an improved-Levin method which is equivalent to the Filon–Clenshaw–Curtis method. The proof of the equivalence can be found in Xiang [13]. For more details of the Filon–Clenshaw–Curtis method, one can refer to the Refs. [14–19]. The asymptotics and evaluation of the oscillatory Bessel Hilbert transform $\fint_0^\infty \frac{f(x)}{x-\tau} J_m(\omega x) dx$ are studied in Xu et al. [20].

The method in this paper is constructed by replacing f by its Lagrange interpolation polynomial at the N Clenshaw–Curtis points and the singular point τ. Furthermore we use a special Hermite interpolation matching the values of its first s derivatives at the endpoints. In addition, we apply this quadrature rule to evaluate the oscillatory integral with logarithmic singularities (see (3.1)). Compared with the methods of [5,15], our methods only take account of the interpolation at the singular point additionally in the case of $s = 0$, while the convergence rates on ω are improved significantly. Moreover, these quadrature rules can also be implemented efficiently with $O(N\log(N))$ operations. It is worth stating that replacing f by the interpolation polynomial at Clenshaw–Curtis points is very common in the approximation of oscillatory integrals [14,15,7,5] which results in a simple method and is not very restrictive for f.

This paper is organized as follows: in Section 2 we describe our improved algorithm for approximating (1.1). In Section 3, we propose a method to compute the integral (3.1) based on the quadrature rule presented in Section 2. In Section 4, we give the convergence rates on ω. The validity of these methods has been demonstrated by several numerical examples in Section 5.

2. Description of the improved algorithm

In the first place, we consider the situation of $\tau \notin \mathbf{X}_N := \{\cos(\frac{j\pi}{N-1})\}_{j=1}^{N-2}$. Suppose $P_{N+2s}(x) = \sum_{j=0}^{N+2s} a_j T_j(x)$ is a Hermite interpolation polynomial of degree $N+2s$ for a non-negative integer s, such that

$$P_{N+2s}(z) = f(z), \quad z \in \mathbf{X}_N \cup \{\tau\}, \qquad P_{N+2s}^{(j)}(\pm 1) = f^{(j)}(\pm 1), \quad j = 0, 1, \ldots, s. \tag{2.1}$$