Error estimate for the numerical solution of fractional reaction–subdiffusion process based on a meshless method

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ABSTRACT

In this paper a numerical technique based on a meshless method is proposed for solving the time fractional reaction–subdiffusion equation. Firstly, we obtain a time discrete scheme based on a finite difference scheme, then we use the meshless Galerkin method, to approximate the spatial derivatives and obtain a full discrete scheme. In the proposed scheme, some integrals appear over the boundary and the domain of problem which will be approximated using Gauss–Legendre quadrature rule. Then, we prove that the time discrete scheme is unconditionally stable and convergent using the energy method. We show convergence order of the time discrete scheme is $\mathcal{O}(\tau^\gamma)$. The aim of this paper is to obtain an error estimate and to show convergence for the meshless Galerkin method based on the radial basis functions. Numerical examples confirm the efficiency and accuracy of the proposed scheme.

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1. Introduction

In recent years there has been a growing interest in the field of fractional calculus [1–4]. Fractional differential equations have attracted increasing attention because they have applications in various fields of science and engineering [5]. Many phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e., the theory of derivatives and integrals of fractional order. Some of the most recent applications are given in the book of Oldham and Spanier [3], the book of Podlubny [4] and the papers of Metzler and Klafter [6], Bagley and Torvik [7]. Many considerable works on the theoretical analysis [8,9] have been carried on, but analytic solutions of most fractional differential equations cannot be obtained explicitly. So many authors have resorted to numerical solution strategies based on convergence and stability analysis [10,11,5,12–16]. Liu [17–19] has carried on so many works on the finite difference method of fractional differential equations. Also we refer the interested reader to [20] for an investigation on the numerical solution of fractional partial differential equations.

There are several definitions of a fractional derivative of order $\alpha > 0$ [3]. The two most commonly used are the Riemann–Liouville and Caputo definitions. The difference between the two definitions is in the order of evaluation [21].

We start with recalling the essentials of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. We give some basic definitions and properties of the fractional calculus theory.

Definition 1 ([4]). For $\mu \in \mathbb{R}$ and $x > 0$, a real function $f(x)$ is said to be in the space C_μ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C(0, \infty)$, and for $m \in \mathbb{N}$ it is said to be in the space C_μ^m if $f^m \in C_\mu$.

Definition 2 ([4]). The Riemann–Liouville fractional integral operator of order $\alpha > 0$ for a function $f(x) \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \, x > 0, \qquad J^0 f(x) = f(x).$$

Also we have the following properties

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \qquad J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \qquad J^\alpha x^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} x^{\alpha+\gamma}.$$

Definition 3 ([4]). Let m be the smallest integer that exceeds α, the Caputo and Riemann–Liouville fractional derivatives of order $\alpha > 0$ are defined as, respectively

$$
{}_0^C D_t^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(m - \alpha)} \displaystyle\int_0^x (x - t)^{m-\alpha-1} \dfrac{d^m f(x)}{dx^m}\bigg|_{x=t} dt, & m - 1 < \alpha < m, \, m \in \mathbb{N}, \\ \dfrac{d^m f(x)}{dx^m}, & m = \alpha. \end{cases}
\tag{1.1}
$$

$$
{}_0 D_t^\alpha f(x) = \begin{cases} \dfrac{d^m}{dx^m} \dfrac{1}{\Gamma(m - \alpha)} \displaystyle\int_0^x (x - t)^{m-\alpha-1} f(t) dt, & m - 1 < \alpha < m, \, m \in \mathbb{N}, \\ \dfrac{d^m f(x)}{dx^m}, & m = \alpha. \end{cases}
\tag{1.2}
$$

Due mainly to the works of K. B. Oldham and his co-authors [22–29], electrochemistry is one of those fields in which fractional-order integrals and derivatives have a strong position and yield practical results. Although the idea of using a half-order fractional integral of current, ${}_0 D_t^{-\frac{1}{2}} i(t)$, can be found also in the works of other authors, it was the paper by K. B. Oldham [26] which definitely opened a new direction in the methods of electrochemistry called semi-integral electroanalysis, later by semidifferential electroanalysis suggested by M. Goto and D. Ishii [30]. One of the important subjects for study in electrochemistry is the determination of the concentration of analyzed electroactive species near the electrode surface. The method suggested by K. B. Oldham and J. Spanier [29] allows, under certain conditions, replacement of a problem for the diffusion equation by a relationship on the boundary (electrode surface). Based on this idea, K. B. Oldham [26] suggested the utilization in experiment the characteristic described by the function

$$m(t) = {}_0 D_t^{-\frac{1}{2}} i(t),$$

which is the fractional integral of the current $i(t)$, as the observed function, whose values can be obtained by measurement. Then the subject of main interest, the surface concentration $C_s(t)$ of the electroactive species, can be evaluated as

$$C_s(t) = C_0 - k_0 D_t^{-\frac{1}{2}} i(t),$$
(1.3)

where k is a certain constant described below, and C_0 is the uniform concentration of the electroactive species throughout the electrolytic medium at the initial equilibrium situation characterized by a constant potential, at which no electrochemical reaction of the considered species is possible. The relationship (1.3) was obtained by considering the following problem for a classical diffusion equation [24]

$$\frac{\partial C(x, t)}{\partial t} = D_* \left[\frac{\partial^2 C(x, t)}{\partial x^2} \right], \quad 0 < x < \infty, \, t > 0,$$

$$C(\infty, 0) = C_{0,}, \qquad C(x, 0) = C_0,$$
(1.4)

$$D_* \left[\frac{\partial C(x, t)}{\partial t} \right]_{x=0} = \frac{i(t)}{n \mathcal{A} \mathcal{F}},$$

where D_* is the diffusion coefficient, $\mathcal{A}$ is the electrode area, $\mathcal{F}$ is Faraday's constant and n is the number of electrons involved in the reaction, the constant k in (1.3) is expressed as

$$k = \frac{1}{n \mathcal{A} \mathcal{F} \sqrt{D_*}}.$$