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Letter to the editor

Gram matrix of Bernstein basis: Properties and applications

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ABSTRACT

This note presents explicit expressions for the inverses of the Gram matrix of the Bernstein basis and its principal submatrices, by taking the advantages of the transformations between the Bernstein basis and the constrained dual Bernstein basis. Using the symmetry property, fast calculation of these matrices and their inverses is achieved. Significant improvements are obtained for applications including polynomial approximation of functions and degree reduction of Bézier curves.

1. Gram matrix of Bernstein basis

Let $B_i^m(t) = \binom{m}{i} t^i(1-t)^{m-i}$, $i = 0, \dots, m$, be the standard Bernstein polynomials of degree m, and define an $(m+1) \times (n+1)$ matrix $G_{mn} = (g_{ij}^{mn})_{0\le i\le m,\ 0\le j\le n}$ with entries

$$g_{ij}^{mn} := \langle B_i^m, B_j^n\rangle = \int_0^1 B_i^m(t)B_j^n(t)\mathrm{d}t = \frac{1}{m+n+1}\binom{m}{i}\binom{n}{j}\Big/\binom{m+n}{i+j}. \tag{1}$$

When $n = m$, $G_{mm} = (g_{ij}^{mm})_{0\le i,j\le m}$ is called the *Gram matrix* of the Bernstein basis $\{B_0^m, \dots, B_m^m\}$. It is well-known that the Gram matrix is symmetric positive definite [1].

When a function is approximated in the least-squares minimization by a Bézier curve represented as a linear combination of control points and Bernstein polynomials, we have to solve a linear system of normal equations, where the inversion of a matrix is unavoidable. In addition, in the field of constrained degree reduction, many methods [2–4] suffer from this limitation. An alternative is to employ orthogonal polynomials or dual polynomials, so that the linear system is trivially solved and the control points are explicitly obtained.

Denote by $\Pi_m^{(k,l)} = \mathrm{span}\{B_k^m, \dots, B_{m-l}^m\}$ $(k, l \ge 0, k+l \le m)$ the space of all polynomials defined on $[0, 1]$ of degree at most m, whose derivatives of order $\le k-1$ at $t = 0$ and of order $\le l-1$ at $t = 1$ vanish. The constrained dual Bernstein basis of degree m, $D_k^{(m,k,l)}, \dots, D_{m-l}^{(m,k,l)} \in \Pi_m^{(k,l)}$, satisfies $\langle D_i^{(m,k,l)}, B_j^m\rangle = \delta_{ij}$ for $i, j = k, \dots, m-l$; see [5–8] for more details.

Throughout the paper, we assume $k, l \ge 0$ and $k+l \le m$, and we denote by $G_{mm}^{kl} = (g_{ij}^{mm})_{k\le i,j\le m-l}$ the principal submatrix of the Gram matrix $G_{mm} = (g_{ij}^{mm})_{0\le i,j\le m}$. Obviously, $G_{mm}^{kl} = G_{mm}$ when $k = l = 0$.

2. Properties

The following two lemmas indicate the transformation matrices between the Bernstein basis and the constrained dual Bernstein basis of $\Pi_m^{(k,l)}$. Especially, it is important to observe from Lemma 1 that G_{mm}^{kl} is exactly the transformation matrix from the constrained dual Bernstein basis to the Bernstein basis.

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Lemma 1. $(B_k^m, \ldots, B_{m-l}^m)^T = G_{mm}^{kl} \times (D_k^{(m,k,l)}, \ldots, D_{m-l}^{(m,k,l)})^T$, *where* $G_{mm}^{kl} = (g_{ij}^{mm})_{k\le i,j\le m-l}$ *with entries expressed in* (1).

Proof. Let $B_i^m(t) = \sum_{j=k}^{m-l} a_{ij} D_j^{(m,k,l)}(t),\ i = k, \ldots, m-l$. Then $a_{ij} = \langle B_i^m, B_j^m\rangle = g_{ij}^{mm}$ follows from $\langle D_\ell^{(m,k,l)}, B_j^m\rangle = \delta_{\ell j}$, similar to the proof of [9, Lemma 2.1]. □

Lemma 2 ([7,9]). $(D_k^{(m,k,l)}, \ldots, D_{m-l}^{(m,k,l)})^T = C_{mm}^{kl} \times (B_k^m, \ldots, B_{m-l}^m)^T$, *where* $C_{mm}^{kl} = (c_{ij}(m,k,l))_{k\le i,j\le m-l}$ *whose entries (abbreviated as* c_{ij}*) satisfy the following recurrence relation. The entries in the first row are (where* $j = k, \ldots, m-l$*)*

$$c_{kj} = (-1)^{j-k}(2k+1)\binom{m}{k}^{-1}\binom{m}{j}^{-1}\binom{m-k-l}{j-k}\binom{m+k-l+1}{2k+1}\binom{m+k+l+1}{k+j+1}. \tag{2}$$

The entries in the row with row-index $i+1 = k+1, \ldots, m-l$ *are (where* $j = k, \ldots, m-l$*)*

$$c_{i+1,j} = \frac{1}{A(i)}[2(i-j)(i+j-m)c_{ij} + B(j)c_{i,j-1} + A(j)c_{i,j+1} - B(i)c_{i-1,j}] \tag{3}$$

with $A(u) := (u-m)(u-k+1)(u+k+1)/(u+1)$, $B(u) := u(u-m-l-1)(u-m+l-1)/(u-m-1)$, *and by convention* $c_{ij} := 0$ *if* i *or* $j \notin \{k, \ldots, m-l\}$.

Remark 1. Besides the recursive scheme to compute the coefficients $c_{ij}(m,k,l)$, explicit formulas are also presented in [7]. The recursive scheme enables more efficient computation.

Proposition 1. G_{mm}^{kl} *and* C_{mm}^{kl} *have the following properties:*

(a) $(G_{mm}^{kl})^{-1} = C_{mm}^{kl},\ (C_{mm}^{kl})^{-1} = G_{mm}^{kl}$;
(b) G_{mm}^{kl} *and* C_{mm}^{kl} *are symmetric positive definite;*
(c) *When* $k = l$, G_{mm}^{kl} *and* C_{mm}^{kl} *are persymmetric. (Note: a matrix is called* ***persymmetric*** *if it is symmetric with respect to the minor diagonal.)*

Proof. (a) follows immediately from Lemmas 1 and 2, and (b) can be verified from the fact that G_{mm} is symmetric positive definite [1,3].

To prove (c), let

$$\tilde{I} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

be an $(m-k-l+1)\times(m-k-l+1)$ permutation matrix, and let

$$\tilde{G}_{mm}^{kl} := (\tilde{I}G_{mm}^{kl}\tilde{I})^T = \tilde{I}G_{mm}^{kl}\tilde{I}, \qquad \tilde{C}_{mm}^{kl} := (\tilde{I}C_{mm}^{kl}\tilde{I})^T = \tilde{I}C_{mm}^{kl}\tilde{I}.$$

Since $k = l$ and $g_{m-j,m-i}^{mm} = g_{ij}^{mm}$ hold for all $i,j = k, \ldots, m-l$, G_{mm}^{kl} is persymmetric, which means $\tilde{G}_{mm}^{kl} = G_{mm}^{kl}$. Then we can derive

$$\tilde{I}C_{mm}^{kl}\tilde{I} = \tilde{I}(G_{mm}^{kl})^{-1}\tilde{I} = (\tilde{G}_{mm}^{kl})^{-1} = (G_{mm}^{kl})^{-1} = C_{mm}^{kl} \Longrightarrow \tilde{C}_{mm}^{kl} = C_{mm}^{kl}$$

which means that C_{mm}^{kl} is also persymmetric. □

Remark 2. By using the explicit formulas for coefficients $c_{ij}(m,k,l)$ (cf. [7, Eq. (2.8)]), it is easy to observe that the matrix C_{mm}^{kl} is symmetric and is also persymmetric when $k = l$. Here, we prove them from the fact that G_{mm}^{kl} and C_{mm}^{kl} are related as the transformation matrices between the Bernstein basis and the constrained dual Bernstein basis. Additionally, we show that C_{mm}^{kl} is positive definite.

Now, we propose fast calculation of G_{mm}^{kl} and C_{mm}^{kl}, using the symmetry property.

Proposition 2. *For* $G_{mm}^{kl} = (g_{ij}^{mm})_{k\le i,j\le m-l}$, *only some entries are calculated by* (1) *and the others are assigned by symmetry. More precisely,*

Case 1 *(*$k = l$*)*

1.1 *Calculate* g_{ij}^{mm} *for* $i = k, \ldots, \lfloor \frac{m}{2} \rfloor,\ j = i, \ldots, m-i$;
1.2 *Set* $g_{ji}^{mm} = g_{ij}^{mm}$ *for* $i = k, \ldots, \lfloor \frac{m}{2} \rfloor,\ j = i+1, \ldots, m-i$;
1.3 *Set* $g_{m-j,m-i}^{mm} = g_{ij}^{mm}$ *for* $i = k, \ldots, m-l-1,\ j = k, \ldots, m-i-1$.