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Preconditioners for regularized saddle point problems with an application for heterogeneous Darcy flow problems

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ABSTRACT

Saddle point problems arise in the modeling of many important practical situations. Preconditioners for the corresponding matrices of block-triangular form, based on coupled inner–outer iteration methods, are analyzed and applied to a Darcy flow problem, possibly with strong heterogeneity and non-symmetric saddle point systems. Using proper regularized forms of the given matrix and its preconditioner, it is shown that the eigenvalues cluster about one or two points on the real axis for large values of the regularization parameters and that eigenvalue bounds do not depend on this variation. Therefore, just two outer iterations can suffice. On the other hand, several iterations may be needed to solve the inner iteration systems.

1. Introduction

Saddle point problems appear in the study of various physical phenomena such as fluid flow, where a pair of variables is coupled via some differential operators. Examples include second or fourth order elliptic problems, which can be reformulated as a coupled system by the introduction of a new variable, typically the gradient of the potential function. This may give rise to a higher order of accuracy for the new variable and put forth a more efficient solution method in some cases, for instance, strongly variable coefficient problems. Such mixed variable formulations of second order problems do exhibit the favorable property that mass is conserved locally. However, these problems are often highly ill-conditioned due to the presence of some coefficients such as a diffusion coefficient, taking widely varying values in the given domain of definition. The construction of robust and efficient preconditioners for solving such problems is of primary concern in this paper. We also show how non-symmetric problems, including the ones with singular or indefinite pivot (1, 1) block matrix, can be preconditioned. In general, the most efficient way to solve saddle point matrix equations is by means of iterative solution methods, for example, see [1–5].

In order to ensure a fast and robust solution method, saddle point problems must be properly preconditioned. Commonly used methods (see [6–11]) are based on approximate block factorization and lead to a form containing a Schur complement matrix, which must be approximated. This may cause complications as the Schur complement is a full matrix and the evaluation of the corresponding residuals may need many inner iterations for determining actions of arising inverses of ill-conditioned matrices and slow rates of convergence. One can find the details of the use of elementwise constructed

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Schur complements in [12]. In this article, we show that certain preconditioners of block tridiagonal form are essentially free of Schur complement matrices, and can behave very efficiently. The methods involve some regularized form of the given matrix or of its preconditioner and are based on coupled inner–outer iteration methods. The idea is to use an efficient regularization method and block matrix factorization which gives very few, typically only 2–4 outer iterations. The block-diagonal matrices arising in the preconditioner are solved by inner iterations. The regularization used in this process leads to better conditioned inner systems which can ensure faster and more robust solutions instead of solving the reduced Schur complement matrix itself.

Other arguments for using a Schur complement free preconditioner can be found in [13]. The inner iteration also needs efficient preconditioners. The construction of such a preconditioner is a topic by itself and will be shortly discussed.

We first show the conditions for non-singularity of the given matrix and then describe how it can be regularized if the conditions are not met, or if the matrix is nearly singular. Two preconditioners of block-triangular form, one for the unregularized and the other one for the regularized matrix are then presented. As the values of regularization parameter increase, clustering of the eigenvalues of the preconditioned matrix occurs about just one or two points on the real axis. This also holds for non-symmetric saddle point problems. This result is shown algebraically for the given finite element matrices and then, in an alternative way, shown to hold by using relations between the corresponding operator pairs, namely for the preconditioning, and the given operators. This result indicates a mesh-independent rate of convergence. The clustering of eigenvalues implies that a fast superlinear rate of convergence takes place after some initial stage.

The details of similar clustering results can be found in the text [4] and papers [14,15]. The major contribution of the present paper is to show these results using short proofs as well as proofs based on operator pair settings. We emphasize that the results for the regularized matrices and their applications are new.

The remainder of the paper is composed as follows. In Section 2, we show conditions for the non-singularity of the given matrix in saddle point form and introduce a regularization for it if necessary. Then the corresponding preconditioned system is analyzed. The regularization used to handle singular systems corresponds actually to the case when the so called LBB (inf–sup) stability condition is violated, for example, see [16] for an earlier presentation of this approach. The preconditioner involves in general two, but similar, parameters, where one is used for the regularized form of the pivot $(1, 1)$ block matrix and the inverse of one to make the $(2, 2)$ block non-zero, but not too big, as it is a perturbation of the given matrix. As the values of the parameters increase, the eigenvalues cluster about one or two points, which in the limit gives just two or three outer iterations for the method.

In Section 3 the corresponding results for the differential operator pairs are presented. The choice of the weight matrix used in the regularization to obtain a better conditioned pivot block matrix is discussed here. This choice depends on the given problem and can be crucial for the efficiency of the method. The results show that mesh-independence and compact perturbation properties hold, that is, the preconditioned matrix is a compact perturbation of identity. Furthermore, it is shown that the eigenvalue bounds for the preconditioned matrix depend little on the variation of coefficients in the differential operator and, hence, are efficient for the problems involving heterogeneous materials such as Darcy flow problems.

The solution of the regularized form of the pivot matrix can be obtained by means of inner iterations. It is then crucial to balance the number of inner and outer iterations for the efficiency of the whole method. This topic is discussed in Section 4. Section 5 contains some numerical tests which involve heterogeneous material coefficient problems as well as advection dominated problems and illustrate the practical bearings of the methods. In the last section, we present some concluding remarks.

2. Efficient preconditioning

Unless otherwise stated, we will denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the minimal and maximal eigenvalues, respectively, of a symmetric matrix A. We use the notations $\mathcal{R}(A)$ and $\mathcal{N}(A)$ respectively, for the range and nullspace of a matrix A. Further rank $(A) = \dim\ \mathcal{R}(A)$.

Consider a given, possibly nonsymmetric, real-valued saddle point matrix in the form

$$\mathcal{M} = \begin{bmatrix} M & B^T \\ C & 0 \end{bmatrix},$$

where M is a square matrix of order $n \times n$ and B, C have orders $m \times n$ with $m \leq n$. Here M may be indefinite but we assume that its symmetric part has at least some positive eigenvalues.

First, we give assumptions to ensure that $\mathcal{M}$ is nonsingular and then, if these conditions are not met, we consider a regularization of $\mathcal{M}$ that becomes nonsingular. In both cases, we present efficient block matrix preconditioners, for which a strong clustering of the eigenvalues of the preconditioned matrix takes place.

2.1. Preconditioning for a nonsingular saddle point matrix

The construction of the preconditioner will be based on the following assumption. Recall that $\text{rank}(A) = \dim\ \mathcal{R}(A)$. If A has order $m \times n$, $m \leq n$, then $\text{rank}(A) \leq m$ and $\dim\ \mathcal{N}(A) = n - \text{rank}(A) \geq n - m$.