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Detecting complete and joint mixability

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ABSTRACT

We introduce the Mixability Detection Procedure (MDP) to check whether a set of d distribution functions is jointly mixable at a given confidence level. The procedure is based on newly established results regarding the convergence rate of the minimal variance problem within the class of joint distribution functions with given marginals. The MDP is able to detect the complete mixability of an arbitrary set of distributions, even in those cases not covered by theoretical results. Stress-tests against borderline cases show that the MDP is fast and reliable.

1. Introduction and motivation of the paper

The definition of complete mixability for a univariate distribution has first been given in [1] and then extended to an arbitrary set of distributions in [2].

Definition 1.1 (*Wang and Wang [1]*). A univariate distribution function F is called *d-completely mixable* (d-CM) if there exist d random variables $X_1, \ldots, X_d$ identically distributed as F having constant sum a.s., that is satisfying

$$\mathbb{P}(X_1 + \cdots + X_d = dc) = 1,$$

for some $c \in \mathbb{R}$.

Definition 1.2 (*Wang et al. [2]*). The d univariate distribution functions $F_1, \ldots, F_d$ are said to be *jointly mixable* (JM) if there exist d random variables $X_1, \ldots, X_d$ such that $X_j \stackrel{d}{=} F_j$, $1 \le j \le d$, and

$$\mathbb{P}(X_1 + \cdots + X_d = C) = 1,$$

for some $C \in \mathbb{R}$.

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It is straightforward that, if F in Definition 1.1 has finite first moment μ, then $c = \mu$, and if each F_j in Definition 1.2 has finite first moment μ_j, then $C = \sum_{j=1}^{d} \mu_j$. The concept of *risks with a constant sum* goes back to [3], where the complete mixability of a set of uniform distributions was showed. The same notion appears in [4], [5, Section 8.3.1] and [6] in the context of variance minimization or as the safest aggregate risk, with a focus on random variables.

The notions of complete and joint mixability have recently gathered a lot of interest since they are related to the existence of a least element with respect to convex order within the set

$$\mathfrak{S}(F_1, \ldots, F_d) := \{X_1 + \cdots + X_d : X_j \stackrel{d}{=} F_j,\ 1 \le j \le d\}$$

consisting of all sums of random variables with given marginal distributions $F_1, \ldots, F_d$. In general the characterization of $\mathfrak{S}(F_1, \ldots, F_d)$: is known to be an open question for $d \ge 2$ (see [7]); is equivalent to the study of joint mixability for $d \ge 3$, by simply observing that $S + C \in \mathfrak{S}(F_1, \ldots, F_d)$ for some $C \in \mathbb{R}$ is equivalent to $F_1, \ldots, F_d, \tilde{F}_S$ are JM, where $\tilde{F}_S$ is the distribution of $-S$. Recall that a random variable X is smaller than Y in convex order, denoted by $X \le_{cx} Y$, if $\mathbb{E}[f(X)] \le \mathbb{E}[f(Y)]$ for all convex functions f such that the expectations exist. When two random variables have the same mean, as within the set $\mathfrak{S}(F_1, \ldots, F_d)$, convex order is equivalent to *increasing convex* order (also known as *stop-loss* order) as defined in [5].

Let U be a U(0, 1) random variable. It is well known (see for instance [8]) that the greatest element wrt convex order in $\mathfrak{S}(F_1, \ldots, F_d)$ is given by the *comonotonic* sum $F_1^{-1}(U) + \cdots + F_d^{-1}(U)$, where

$$F_j^{-1}(p) = \begin{cases} \inf\{x \in \mathbb{R} : F_j(x) > p\}, & \text{if } p \in [0, 1), \\ \inf\{x \in \mathbb{R} : F_j(x) \ge 1\}, & \text{if } p = 1, \end{cases}$$

is the generalized inverse (or quantile function) of F_j, $1 \le j \le d$; see [9] for more details on the concept of comonotonicity and several related results. In fact, we have that

$$X_1 + \cdots + X_d \le_{cx} F_1^{-1}(U) + \cdots + F_d^{-1}(U),$$

for any $X_j \stackrel{d}{=} F_j$, $1 \le j \le d$. When there are only two random variables, i.e. $d = 2$, the $\le_{cx}$-least element in $\mathfrak{S}(F_1, \ldots, F_d)$ is known to be the *countermonotonic* sum $F_1^{-1}(U) + F_2^{-1}(1 - U)$, i.e.

$$F_1^{-1}(U) + F_2^{-1}(1 - U) \le_{cx} X_1 + X_2,$$

for any $X_1 \stackrel{d}{=} F_1$ and $X_2 \stackrel{d}{=} F_2$. When $d > 2$, the problem of determining the existence of a least element in $\mathfrak{S}(F_1, \ldots, F_d)$ is much more complicated as the notion of a countermonotonic sum with given marginals cannot be generalized to higher dimensions; this was studied in [10], and we refer to [7,11] for recent discussions.

It is a trivial observation that if $F_1, \ldots, F_d$ have finite means $\mu_1, \ldots, \mu_d$ and are JM, the least element in $\mathfrak{S}(F_1, \ldots, F_d)$ is given by $\mu_1 + \cdots + \mu_d$, i.e.

$$\mu_1 + \cdots + \mu_d \le_{cx} X_1 + \cdots + X_d,$$

for any $X_j \stackrel{d}{=} F_j$, $1 \le j \le d$. Existence of $\le_{cx}$-least elements on sums and the corresponding conditions of complete/joint mixability are involved in a variety of optimization problems in the theory of optimal couplings, as for example:

(i) Assume that $F_1, \ldots, F_d$ have finite first moment $\mu_1, \ldots, \mu_d$ with $\mu = \sum_{j=1}^{d} \mu_j$. For a (strictly) convex function $f : \mathbb{R} \to \mathbb{R}$, we have by Jensen's inequality that

$$\inf\{\mathbb{E}\left[f(X_1 + \cdots + X_d)\right]; X_j \stackrel{d}{=} F_j,\ 1 \le j \le d\} \ge f(\mu), \tag{1.1}$$

and equality holds if (and only if) $F_1, \ldots, F_d$ are JM.

(ii) Assume that $F_1, \ldots, F_d$ are continuous and have finite first moment. Let $X_j \stackrel{d}{=} F_j,\ 1 \le j \le d$, and, for $a \in [0, 1]$, define the function

$$\Psi(a) = \sum_{j=1}^{d} \mathbb{E}[X_j | X_j \ge F_j^{-1}(a)].$$

For any $s \ge \mu$, we have

$$M(s) = \sup\{P(X_1 + \cdots + X_d \ge s); X_j \sim F_j,\ 1 \le j \le d\} \le 1 - \Psi^{-}(s), \tag{1.2}$$

where $\Psi^{-}(s) = \sup\{t \in [0, 1] : \Psi(t) \le s\}$ and the sup is attained if and only if the conditional distributions of $(X_j | X_j \ge F_j^{-1}(\Psi^{-}(s)))$ are JM.

Problems (1.1) and (1.2) have relevant applications in quantitative risk management, where they are needed to assess the model risk associated to the computation of capital charges for a portfolio of losses for regulatory issues. For instance, problem (1.1) is related to the computation of bounds on the expected value of a supermodular function [1,12] and on the expected shortfall of a sum of random variables [13]. When f in (1.1) is chosen as $f(x) = (x - \mu)^2$, (1.1) becomes a variance minimization problem, which is fundamental in variance reduction and simulation; see for example [14]. Problem (1.2), as