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Hermitian approximation of the spherical divergence on the Cubed-Sphere

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ABSTRACT

Previous work (Croisille, 2013) showed that the Cubed-Sphere grid offers a suitable discrete framework for extending Hermitian compact operators (Collatz, 1960) to the spherical setup. In this paper we further investigate the design of high-order accurate approximations of spherical differential operators on the Cubed-Sphere with an emphasis on the spherical divergence of a tangent vector field. The basic principle of this approximation relies on evaluating pointwise Hermitian derivatives along a series of great circles covering the sphere. Several test-cases demonstrate the very good accuracy of the approximate spherical divergence calculated with the new scheme.

1. Introduction

Recently accurate approximations of partial differential equations on the sphere have become increasingly demanded. Many fields in physics beyond Global Climatology Modeling, where calculations on the rotating earth is a core practice, now require a significant use of the spherical context. These include

- *Medical imaging*: The inverse Radon transform consists in approximating spectral data on the sphere in a fast and accurate manner [1].
- *Gravitational field analysis*: Specialized satellite data (from the COBE satellite) must be thoroughly worked out by interpolation and approximation [2].
- *Celestial data analysis*: Data given on the celestial sphere also require an accurate treatment [3].

In this work the principle behind approximating spherical differential operators [4] is further applied to the spherical divergence. Our compact approximation is performed on The Cubed-Sphere [5–7]. It uses a methodology based on the following steps [4]:

- Interpolation of data on a series of great circles using the fact that coordinate lines are great circles sections.
- Calculation of Hermitian finite-difference derivatives along these great circles. This step is purely one-dimensional and periodic, only requiring tridiagonal (periodic) linear systems to be solved.
- Reconstruction of spherical differential operator on each panel of the Cubed-Sphere using a suitable change of variables.

The expected fourth-order accuracy inherited from Hermitian approximations was observed. This is particularly important for calculations where second order accuracy is not sufficient. It is the case for example for calculating complex fluid

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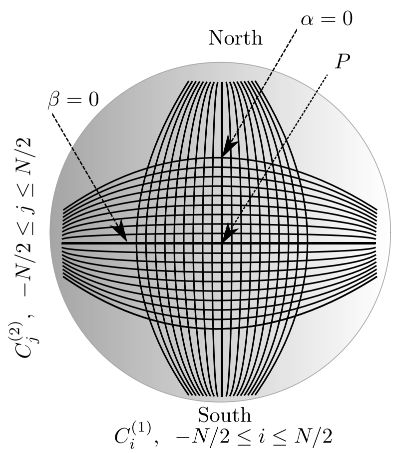

Fig. 1. The points of the Cubed Sphere are obtained as intersections of two families of great circles. The circles in *vertical* (South–North) position are labeled $C_i^{(1)}, \ -N/2 \leq i \leq N/2$. The circles in *horizontal* (East–West) position are labeled $C_j^{(2)}, -N/2 \leq j \leq N/2$. The isovalues circles $\alpha = 0$ and $\beta = 0$ are indicated. The point P serves as the center of each family of great circles.

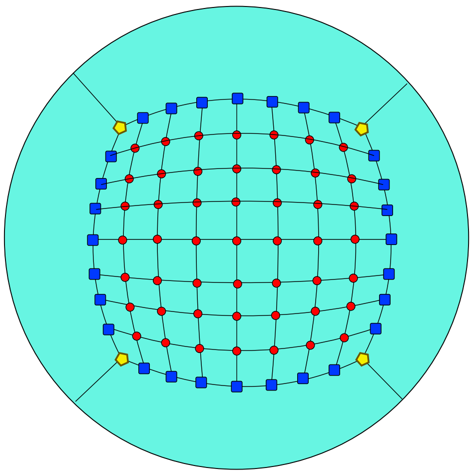

Fig. 2. The points of a typical panel of the Cubed-Sphere are classified in three categories: (i) Circles correspond to *internal* points; (ii) Squares correspond to *edge* points; (iii) Pentagons correspond to *corner* points.

flow patterns, such as turbulent flows, or quasi incompressible flows. Other important examples can be found in Global Climatology Modeling, where several series of test-cases have been suggested [8–10]. These tests offer an excellent platform for assessing the efficiency of numerical schemes for PDE's approximations on the sphere.

The outline of the paper is as follows. In Section 2 the fundamental principle for Hermitian approximation is recalled. In Section 3 the calculation of the spherical gradient [4] is recalled. Section 4 is devoted to calculating the spherical divergence of a tangential vector field. In Section 5 several results showing numerical evidence of the fourth-order accuracy are presented. After the conclusion (Section 6), standard geometric notation used throughout the paper is recalled in the Appendix.

2. The Cubed-Sphere grid

2.1. Panels and geodesic coordinate lines

In this section we present the Cubed-Sphere grid described in [11]. This grid is commonly used for simulation of the dynamics of the atmosphere. Refer to [12,13] and the references therein. In the sequel we emphasize why this grid is well adapted to calculate high order discrete differential operators in the Hermitian fashion. Note moreover that all our calculations are done on the sphere itself and not on the faces of the cube in which the sphere is embedded.

The design of the Cubed-Sphere is based on great circles, i.e. geodesic lines of the sphere $\mathbb{S}^2$ equipped with the standard metric. Consider the two sets of great circles called $C_i^{(1)}$ and $C_j^{(2)}$ shown in Fig. 1. These two sets are constructed as follows. The set $C_i^{(1)}$ consists of a collection of great circles in *South–North* (meridional) position, passing through the South and North poles. A point P is selected on one of these great circles and will serve as the center of the grid described in what follows.