Journal of Computational and Applied Mathematics 280 (2015) 248–264

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Convergence and almost sure exponential stability of implicit numerical methods for a class of highly nonlinear neutral stochastic differential equations with constant delay

Marija Milošević

University of Niš, Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia

HIGHLIGHTS

- This manuscript represents the continuation of the previous work of the author, related to a class of highly nonlinear neutral stochastic differential equations with time-dependent delay.
- Main results of this manuscript are convergence in probability and almost sure exponential stability of the backward Euler approximate solution for a class of stochastic differential equations with constant delay.
- Conditions under which both, the exact and approximate solutions share the property of the almost sure exponential stability are revealed.
- This paper also illustrates that, in some cases, stochastic differential equations with constant delay should be studied separately from those with time-dependent delay. In that sense, it should be stressed that results of this paper are obtained under weaker conditions comparing to those which would be obtained by replacing the time-dependent delay by the constant delay.

ARTICLE INFO

Article history:
Received 5 February 2014
Received in revised form 1 December 2014

MSC:
60H10

Keywords:
Neutral stochastic differential delay equations
Nonlinear growth conditions
One-sided Lipschitz condition
Backward Euler method
Global a.s. asymptotic exponential stability

ABSTRACT

This paper can be regarded as the continuation of the work contained in papers Milošević (2011, 2013). At the same time, it represents the extension of the paper Wu et al. (2010). In this paper, the one-sided Lipschitz condition is employed in the context of the backward Euler method, for a class of neutral stochastic differential equations with constant delay. Sufficient conditions for this method to be well defined are revealed. Under certain nonlinear growth conditions, the convergence in probability is established for the continuous forward–backward Euler method, as well as for the discrete backward Euler method. Additionally, it is proved that the discrete backward Euler equilibrium solution is globally a.s. asymptotically exponentially stable, without requiring for the drift coefficient to satisfy the linear growth condition.

1. Introduction and preliminary results

There is already an extensive literature related to (neutral) stochastic differential delay equations or (neutral) stochastic functional differential equations because of many possibilities for their application. Different qualitative and quantitative properties of solutions to these equations, under linear growth condition on coefficients, are studied, for example, in [1–5].

However, in many cases it is necessary to employ highly nonlinear stochastic differential equations, that is, equations with coefficients which do not satisfy the linear growth condition. We refer the reader to [6–13] and the literature cited therein,

E-mail address: 27marija.milosevic@gmail.com.

http://dx.doi.org/10.1016/j.cam.2014.12.002

where the existence, uniqueness, stability and approximations of solutions are considered, for stochastic differential delay equations and neutral stochastic differential equations with time-dependent delay.

The subject of this paper is consideration of implicit numerical methods for a class of neutral stochastic differential equations with constant delay, under certain nonlinear growth conditions, known as Khasminskii-type conditions. These conditions represent the appropriate modifications of conditions from the paper [11], which deals with neutral stochastic differential equations with time-dependent delay. In [11], the author proved the existence and uniqueness of solution, as well as the convergence in probability of the appropriate Euler–Maruyama solution to the exact solution under these conditions. Also, in [12], global a.s. asymptotic exponential stability of the exact equilibrium solution to the equation of this type is established, without the linear growth condition on the drift coefficient of the equation. By adding the linear growth condition for the drift coefficient, the same type of stability is proved for the Euler–Maruyama equilibrium solution.

Thus, it is natural to investigate under which conditions the exact and the appropriate approximate solutions share same properties. For example, in [14], the author studied global a.s. asymptotic stability of exact and numerical equilibrium solutions for neutral stochastic pantograph equations, under nonlinear growth conditions. In [15], the analysis of implicit numerical methods for ordinary stochastic differential equations with non-globally Lipschitz continuous coefficients is presented. Moreover, in [16,17], more general numerical θ-method for ordinary and delay stochastic differential equations is considered under certain nonlinear growth conditions.

The main motivation for this paper came from [15], where authors proved that the backward Euler equilibrium solution for stochastic differential equations with constant delay is globally a.s. asymptotically exponentially stable, without the linear growth condition on the drift coefficient of the equation. However, the one-sided Lipschitz condition is required in order for the backward Euler method to be well defined. Bearing in mind these results and results from [11,12], it was natural to investigate weather or not the backward Euler method for neutral stochastic differential equations with constant delay can give globally a.s. asymptotically exponentially stable solutions, without the linear growth condition on the drift coefficient. It should be stressed that, in further analysis, the global Lipschitz condition on the coefficient of the equation is not required. However, we will require for the drift coefficient of the equation to satisfy the one-sided Lipschitz condition.

The one-sided Lipschitz condition is successfully exploited in the context of numerical analysis of stochastic differential equations, as one can observe from the papers [15,18–21] and the monograph [22], as well as from the references cited therein.

Contents of this paper are given below.

First, the basic notation and hypotheses which are necessary for proving the main results of this paper will be introduced. Then, in Section 2 we will define the backward Euler method for neutral stochastic differential equation with constant delay. Employing the one-sided Lipschitz condition for the first argument of the drift coefficient of the equation, we will reveal the scope of the step-size for which the backward Euler method is well defined. In Section 3, we will show that the forward–backward and backward Euler solutions converge in probability to the exact solution of the equation. Finally, in Section 4, we will establish the global a.s. asymptotic exponential stability of the backward Euler solution, under slightly stronger assumptions then those from previous sections. It should be stressed that this result will be obtain without the linear growth condition on the drift coefficient.

Before introducing the main results of this paper, we present the basic notation and hypotheses which are necessary for further consideration. Because of the fact that this paper is closely related to papers [11] and [12], these notation and hypotheses are mostly the same as those from these papers. However, they will be introduced here for the consistency of the paper. Thus, the initial assumption is that all random variables and processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), w_2(t), \dots, w_m(t))^T$, $t \geq 0$ be an m-dimensional standard Brownian motion, $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. Let the Euclidean norm be denoted by $|\cdot|$ and, for simplicity, $trace[B^T B] = |B|^2$ for matrix B, where B^T is the transpose of a vector or a matrix.

For a given $\tau > 0$, let $C([-\tau, 0]; R^d)$ be the family of continuous functions φ from $[-\tau, 0]$ to R^d, equipped with the supremum norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. Also, denote by $C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$ the family of bounded, $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^d)$-valued random variables.

The following consideration is related to the neutral stochastic differential equation with constant delay of the form

$$d[x(t) - u(x(t-\tau), t)] = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dw(t), \quad t \geq 0, \tag{1}$$

satisfying the initial condition

$$x_0 = \xi = \{\xi(\theta), \theta \in [-\tau, 0]\}. \tag{2}$$

Coefficients of Eq. (1)

$$f : R^d \times R^d \times R_+ \to R^d, \qquad g : R^d \times R^d \times R_+ \to R^{d\times m}, \qquad u : R^d \times R_+ \to R^d,$$

are all Borel measurable functions and $x(t)$ is a d-dimensional state process. The initial condition ξ is supposed to be a $C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$-valued random variable.

A d-dimensional stochastic process $\{x(t), t \geq -\tau\}$ is said to be a solution to Eq. (1) if it is a.s. continuous, $\mathcal{F}_t$-adapted, $\int_0^\infty |f(x(t), x(t-\tau), t)|dt < \infty$ a.s., $\int_0^\infty |g(x(t), x(t-\tau), t)|^2 dt < \infty$, a.s., $x_0 = \xi$ a.s. and for every $t \geq 0$, the integral form of Eq. (1) holds a.s.